

Super-characters of finite unipotent groups of types B_n , C_n and D_n

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Received 9 September 2005

Available online 9 June 2006

Communicated by Jan Saxl

Abstract

We define and study super-characters (over the complex field) of the classical finite unipotent groups of types B_n , C_n and D_n . Under the assumption that the prime is sufficiently large, we extend the known results for the unitriangular group proved by the first author in the papers: [C.A.M. André, Basic characters of the unitriangular group, *J. Algebra* 175 (1995) 287–319], and [C.A.M. André, Basic characters of the unitriangular group (for arbitrary primes), *Proc. Amer. Math. Soc.* 130 (7) (2002) 1943–1954]. In particular, we prove that every irreducible (complex) character occurs as a constituent of a unique super-character. We also give a combinatorial description of all the irreducible characters of maximum degree.

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Keywords: Finite unipotent group; Symplectic group; Orthogonal group; Irreducible character; Super-character; Positive root; Basic set of positive roots

1. Introduction

Let $p \geq 3$ be a prime number, $q = p^e$ ($e \geq 1$) a power of p and \mathbb{F}_q the finite field with q elements. For a fixed positive integer n , let G denote one of the following classical finite groups

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defined over \mathbb{F}_q : the symplectic group $Sp_{2n}(q)$, the even orthogonal group $O_{2n}(q)$, or the odd orthogonal group $O_{2n+1}(q)$. Throughout the paper, we set $U = G \cap U_m(q)$ where

$$m = \begin{cases} 2n, & \text{if, either } G = Sp_{2n}(q), \text{ or } G = O_{2n}(q), \\ 2n+1, & \text{if } G = O_{2n+1}(q), \end{cases}$$

and where $U_m(q)$ denotes the upper unitriangular group consisting of all unipotent upper-triangular $m \times m$ matrices over \mathbb{F}_q . For convenience, we shall use the notation $USp_{2n}(q)$, $UO_{2n}(q)$ or $UO_{2n+1}(q)$ to refer to the subgroup U of $Sp_{2n}(q)$, $O_{2n}(q)$ or $O_{2n+1}(q)$, respectively. Let J be the $n \times n$ matrix with 1's along the anti-diagonal and 0's elsewhere. Then,

$$\begin{aligned} \text{(i)} \quad USp_{2n}(q) &= \left\{ \begin{pmatrix} x & xz \\ 0 & Jx^{-T}J \end{pmatrix} \in U_{2n}(q) : Jz^T - zJ = 0 \right\}; \\ \text{(ii)} \quad UO_{2n}(q) &= \left\{ \begin{pmatrix} x & xz \\ 0 & Jx^{-T}J \end{pmatrix} \in U_{2n}(q) : Jz^T + zJ = 0 \right\}; \\ \text{(iii)} \quad UO_{2n+1}(q) &= \left\{ \begin{pmatrix} x & xu & xz \\ 0 & 1 & -u^TJ \\ 0 & 0 & Jx^{-T}J \end{pmatrix} \in U_{2n+1}(q) : Jz^T + zJ = -uu^T \right\}. \end{aligned}$$

[We observe that U has order q^{n^2} if $U = USp_{2n}(q)$ or $U = UO_{2n+1}(q)$, and $q^{n(n-1)}$ if $U = UO_{2n}(q)$; hence, U is a Sylow p -subgroup of G (the order of G can be found in [7, p. 75]).]

The main goal of this paper is to define and prove the main properties of the “super-characters” of the group U . Our definition extends the one given previously for the case of the unitriangular group $U_n(q)$ (see the papers [1,3] and also [11]; in the first two papers, the super-characters were called basic characters), and is also given in terms of certain subsets of (positive) roots. Thus, we introduce some notation and recall some elementary facts concerning roots (for the details, we refer to the books [6,7]; see also [10, Chapter 8]). Let T be the maximal torus of G consisting of all diagonal matrices and $\Phi = \Phi(G, T)$ the root system defined by T . By definition, Φ is a subset of the finite abelian group $X(T) = \text{Hom}(T, \mathbb{F}_q^\times)$ consisting of (rational) homomorphisms from T to \mathbb{F}_q^\times ; as usual, we denote by \mathbb{F}_q^\times the multiplicative group of \mathbb{F}_q and we use the additive notation for the operation of the group $X(T)$. The elements of Φ are described as follows. For each $1 \leq i \leq n$, let $\varepsilon_i \in X(T)$ be defined by $\varepsilon_i(t) = t_i$ for all $t \in T$; here, we denote by $t_i \in \mathbb{F}_q^\times$ the (i, i) th entry of the matrix $t \in T$. Then, $\Phi = \Phi^+ \cup \Phi^-$ where $\Phi^- = -\Phi^+$ and

$$\Phi^+ = \begin{cases} \{\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n\} \cup \{2\varepsilon_i : 1 \leq i \leq n\}, & \text{if } G = Sp_{2n}(q), \\ \{\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n\}, & \text{if } G = O_{2n}(q), \\ \{\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n\} \cup \{\varepsilon_i : 1 \leq i \leq n\}, & \text{if } G = O_{2n+1}(q). \end{cases}$$

The roots in Φ^+ are said to be *positive* (and the roots in Φ^- are said to be *negative*).

With Φ^+ we associate the subset of “entries” $\mathcal{E} \subseteq \{(i, j) : -n \leq i, j \leq n\}$ as follows. For any $\alpha \in \Phi^+$, we set

$$\mathcal{E}(\alpha) = \begin{cases} \{(i, j), (-j, -i)\}, & \text{if } \alpha = \varepsilon_i - \varepsilon_j \text{ for } 1 \leq i < j \leq n, \\ \{(i, -j), (j, -i)\}, & \text{if } \alpha = \varepsilon_i + \varepsilon_j \text{ for } 1 \leq i < j \leq n, \\ \{(i, -i)\}, & \text{if } G = Sp_{2n}(q) \text{ and } \alpha = 2\varepsilon_i \text{ for } 1 \leq i \leq n, \\ \{(i, 0), (0, -i)\}, & \text{if } G = O_{2n+1}(q) \text{ and } \alpha = \varepsilon_i \text{ for } 1 \leq i \leq n, \end{cases}$$

and we define

$$\mathcal{E} = \bigcup_{\alpha \in \Phi^+} \mathcal{E}(\alpha).$$

More generally, for each subset Ψ of Φ^+ , we define

$$\mathcal{E}(\Psi) = \bigcup_{\alpha \in \Psi} \mathcal{E}(\alpha).$$

On the other hand, we consider the mirror order \prec on the set $\{0, \pm 1, \dots, \pm n\}$ which is defined as

$$1 \prec 2 \prec \dots \prec n \prec 0 \prec -n \prec \dots \prec -2 \prec -1,$$

and we shall index the rows (from left to right) and columns (from top to bottom) of any $m \times m$ matrix according to this ordering. Hence, the entries of any matrix $x \in U_m(q)$ are indexed by all the pairs $(i, j) \in \mathcal{E}$: for each $(i, j) \in \mathcal{E}$, we shall write $x_{i,j}$ to denote the (i, j) th entry of x (which occurs in the i th row and in the j th column).

For any $\alpha \in \Phi^+$, we define the subgroup U_α of U as follows:

(i) if $\alpha = \varepsilon_i - \varepsilon_j$ for $1 \leq i < j \leq n$, then

$$U_\alpha = \{x \in U : x_{i,k} = 0, i < k < j\};$$

(ii) if $\alpha = \varepsilon_i + \varepsilon_j$ for $1 \leq i < j \leq n$, then

$$U_\alpha = \{x \in U : x_{i,k} = x_{j,l} = 0, i < k \leq n, j < l \leq 0\};$$

(iii) if, either $\alpha = 2\varepsilon_i$ for $1 \leq i \leq n$ (in the case where $U = USp_{2n}(q)$), or $\alpha = \varepsilon_i$ for $1 \leq i \leq n$ (in the case where $U = UO_{2n+1}(q)$), then

$$U_\alpha = \{x \in U : x_{i,k} = 0, i < k \leq n\}.$$

Let $\psi : \mathbb{F}_q \rightarrow \mathbb{C}^\times$ be a non-trivial linear character of the additive group \mathbb{F}_q^+ of \mathbb{F}_q (this character will be kept fixed throughout the paper; moreover, all characters will be taken over the complex field). Then, for any $r \in \mathbb{F}_q^\times$, the mapping $x \mapsto \psi(rx_{i,j})$ defines a linear character $\lambda_{\alpha,r} : U_\alpha \rightarrow \mathbb{C}^\times$ of U_α , and we define the *elementary character* $\xi_{\alpha,r}$ to be the induced character

$$\xi_{\alpha,r} = \lambda_{\alpha,r}^U$$

(see [1,3] for the corresponding definition in the case of the unitriangular group).

As in the case of the unitriangular group, we may define the super-characters of U as certain “reduced” products of elementary characters. Firstly, we define the notion of basic subset of positive roots. To start with, we recall the notion of a basic subset of \mathcal{E} . For any $-n \leq i, j \leq n$, we set

$$r_i(n) = \{(i, k) \in \mathcal{E} : -n \leq k \leq n\}, \quad c_j(n) = \{(k, j) \in \mathcal{E} : -n \leq k \leq n\};$$

we refer to $r_i(n)$ as the i th row of \mathcal{E} , and to $c_j(n)$ as the j th column of \mathcal{E} . A subset $D_{\mathcal{E}} \subseteq \mathcal{E}$ is said to be *basic* if $D_{\mathcal{E}}$ contains at most one entry from each row $r_i(n)$ and at most one entry from each column $c_j(n)$; in other words, $D_{\mathcal{E}} \subseteq \mathcal{E}$ is basic if, and only if,

$$|D_{\mathcal{E}} \cap r_i(n)| \leq 1 \quad \text{and} \quad |D_{\mathcal{E}} \cap c_j(n)| \leq 1, \quad \text{for all } -n \leq i, j \leq n.$$

Finally, we say that $D \subseteq \Phi^+$ is a *basic subset* if, and only if, $\mathcal{E}(D)$ is a basic subset of \mathcal{E} . [We observe that, if $U = U_n(q)$ or if $U = USp_{2n}(q)$, then a subset D of Φ^+ is *basic* if and only if D satisfies the following condition for all $\alpha, \beta \in \Phi^+$: if $\alpha \in D$ and $\alpha - \beta \in \Phi$, then $\beta \notin D$. However, this is not the case if $U = UO_{2n}(q)$ or if $U = UO_{2n+1}(q)$; in fact, in either case, $D = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j\}$, for $1 \leq i < j \leq n$, satisfies that condition but is not a basic subset of roots (for example, $(i, j), (i, -j) \in \mathcal{E}(D)$).]

Given any non-empty basic subset $D \subseteq \Phi^+$ and any map $\phi: D \rightarrow \mathbb{F}_q^\times$, we define the *super-character* $\xi_{D,\phi}$ to be the product

$$\xi_{D,\phi} = \prod_{\alpha \in D} \xi_{\alpha, \phi(\alpha)}.$$

For convenience, if D is the empty subset of Φ^+ , we consider the empty map $\phi: D \rightarrow \mathbb{F}_q^\times$ and we define $\xi_{D,\phi}$ to be the unit character 1_U of U . We now state the main result of this paper (see [1, Theorem 1] and [3, Theorem 1] for the case of the unitriangular group). [Given any finite group G , we denote by $\langle \cdot, \cdot \rangle_G$, or simply by $\langle \cdot, \cdot \rangle$, the Frobenius scalar product on the complex vector space of all class functions defined on G .]

Theorem 1.1. *Let χ be an arbitrary irreducible character of U and assume that $p \geq 2n$. Then, χ is a constituent of a unique super-character of U ; in other words, there exists a unique basic subset D of Φ^+ and a unique map $\phi: D \rightarrow \mathbb{F}_q^\times$ such that $\langle \chi, \xi_{D,\phi} \rangle \neq 0$.*

The assumption $p \geq 2n$ is used in the proof of this result, but we hope that the theorem is also true for small values of p . This assumption is made in order to allow the use of the exponential map. In fact, although our results are primarily concerned with the character theory of a finite group, our proofs use the method of coadjoint orbits introduced by A. Kirillov in the context of nilpotent Lie groups (see [9,16]; see also [17]) and adapted to the case of finite unipotent groups by D. Kazhdan (see [15] or [20, Chapter 7]). In fact, some of our results have an obvious generalization to the case of a nilpotent Lie algebra defined over an arbitrary field (not necessarily finite).

2. Super-characters

For each $(a, b) \in \mathcal{E}$, let $e_{a,b}$ denote the standard elementary $m \times m$ matrix having 1 in the (a, b) th entry and 0's elsewhere, and, for each $\alpha \in \Phi^+$ and each $t \in \mathbb{F}_q$, let $x_\alpha(t)$ denote the element of U defined by

$$x_\alpha(t) = \begin{cases} 1 + t(e_{i,j} - e_{-j,-i}), & \text{if } \alpha = \varepsilon_i - \varepsilon_j, \\ 1 + t(e_{i,-j} + e_{j,-i}), & \text{if } \alpha = \varepsilon_i + \varepsilon_j \text{ and } U = USp_{2n}(q), \\ 1 + t(e_{i,-j} - e_{j,-i}), & \text{if } \alpha = \varepsilon_i + \varepsilon_j \text{ and } U \neq USp_{2n}(q), \\ 1 + te_{i,-i}, & \text{if } U = USp_{2n}(q) \text{ and } \alpha = 2\varepsilon_i, \\ 1 + t(e_{i,0} - e_{0,-i}) - \frac{t^2}{2}e_{i,-i}, & \text{if } U = UO_{2n+1}(q) \text{ and } \alpha = \varepsilon_i. \end{cases}$$

The mapping $t \mapsto x_\alpha(t)$ defines an injective group homomorphism from \mathbb{F}_q^+ to U , and so the image

$$X_\alpha = \{x_\alpha(t): t \in \mathbb{F}_q\}$$

is a subgroup of U ; in the usual terminology, X_α is called the *root subgroup* of U associated with $\alpha \in \Phi^+$. It is well known that, for any fixed ordering of the positive roots, we have

$$U = \prod_{\alpha \in \Phi^+} X_\alpha$$

and that any element $x \in U$ can be written in the form

$$x = \prod_{\alpha \in \Phi^+} x_\alpha(t_\alpha),$$

where $t_\alpha \in \mathbb{F}_q$, for $\alpha \in \Phi^+$, are uniquely determined (see, for example, [7, Corollary 2.5.17]).

Now, we fix an arbitrary root $\alpha \in \Phi^+$ and consider the subgroup U_α of U as defined in the introduction. We define

$$\Phi^+(\alpha) = \{\beta \in \Phi^+: X_\beta \subseteq U_\alpha\}.$$

Hence, $\Phi^+(\alpha) = \Phi^+ - \Psi(\alpha)$ where $\Psi(\alpha)$ is as follows:

(i) if $\alpha = \varepsilon_i - \varepsilon_j$ for $1 \leq i < j \leq n$, then

$$\Psi(\alpha) = \{\varepsilon_i - \varepsilon_k: i < k < j\};$$

(ii) if $\alpha = \varepsilon_i + \varepsilon_j$ for $1 \leq i < j \leq n$, then

$$\Psi(\alpha) = \{\varepsilon_i - \varepsilon_k: i < k \leq n\} \cup \{\varepsilon_j - \varepsilon_l: j < l \leq 0\},$$

where we write $\varepsilon_0 = 0$ (in the case where $U = UO_{2n+1}(q)$);

(iii) if, either $\alpha = 2\varepsilon_i$ for $1 \leq i \leq n$ (in the case where $U = USp_{2n}(q)$), or $\alpha = \varepsilon_i$ for $1 \leq i \leq n$ (in the case where $U = UO_{2n+1}(q)$), then

$$\Psi(\alpha) = \{\varepsilon_i - \varepsilon_k: i < k \leq n\}.$$

It is straightforward to check that

$$U_\alpha = \prod_{\beta \in \Phi^+(\alpha)} X_\beta.$$

In particular, X_α is a subgroup of U_α (in fact, we have $\alpha \in \Phi^+(\alpha)$). Moreover, $\Phi^+(\alpha) - \{\alpha\}$ is a closed subset of Φ^+ (i.e., $\beta + \gamma \in \Phi^+(\alpha) - \{\alpha\}$ whenever $\beta, \gamma \in \Phi^+(\alpha) - \{\alpha\}$ and $\beta + \gamma \in \Phi^+$) and so

$$N_\alpha = \prod_{\beta \in \Phi^+(\alpha) - \{\alpha\}} X_\beta$$

is a subgroup of U_α ; in fact, it is not difficult to prove that N_α is a normal subgroup of U_α and that $U_\alpha = X_\alpha N_\alpha$ is a semidirect product. Thus, every element $x \in U_\alpha$ can be written uniquely in the form $x = x_\alpha(s)z$ where $s \in \mathbb{F}_q$ and $z \in N_\alpha$; we observe that, if $(i, j) \in \mathcal{E}(\alpha)$, then $s \in \mathbb{F}_q$ is the (i, j) th entry of the matrix $x = x_\alpha(s)z \in U_\alpha$. For each $r \in \mathbb{F}_q^\times$, we define $\lambda_{\alpha,r}: U_\alpha \rightarrow \mathbb{C}^\times$ by

$$\lambda_{\alpha,r}(x_\alpha(s)z) = \psi(rs), \quad \text{for all } s \in \mathbb{F}_q \text{ and all } z \in N_\alpha$$

(we note that this coincides with the definition given in the introduction). Since

$$x_\alpha(s)N_\alpha x_\alpha(s')N_\alpha = x_\alpha(s + s')N_\alpha, \quad \text{for all } s, s' \in \mathbb{F}_q,$$

we conclude that $\lambda_{\alpha,r}$ is a linear character of U_α with $N_\alpha \subseteq \ker(\lambda_{\alpha,r})$. We define the *elementary character* $\xi_{\alpha,r}$ of U (associated with α and r) to be the induced character

$$\xi_{\alpha,r} = \lambda_{\alpha,r}^U.$$

For our purposes, it is convenient to partition the set Φ^+ into three subsets

$$\Phi_1^+ = \{\varepsilon_i - \varepsilon_j: 1 \leq i < j \leq n\}, \quad \Phi_2^+ = \{\varepsilon_i + \varepsilon_j: 1 \leq i < j \leq n\}$$

and

$$\Phi_3^+ = \begin{cases} \{2\varepsilon_i: 1 \leq i \leq n\}, & \text{if } U = USp_{2n}(q), \\ \emptyset, & \text{if } U = UO_{2n}(q), \\ \{\varepsilon_i: 1 \leq i \leq n\}, & \text{if } U = UO_{2n+1}(q). \end{cases}$$

It is clear that Φ_1^+ and $\Phi_2^+ \cup \Phi_3^+$ are closed subsets of Φ^+ , hence

$$H = \prod_{\beta \in \Phi_1^+} X_\beta \quad \text{and} \quad A = \prod_{\beta \in \Phi_2^+ \cup \Phi_3^+} X_\beta$$

are subgroups of U . We observe that H is naturally isomorphic to the unitriangular group $U_n(q)$ via the mapping $x \mapsto \begin{pmatrix} x & 0 \\ 0 & Jx^{-T}J \end{pmatrix}$ for $x \in U_n(q)$, and that A is a normal subgroup of U ; hence, we have a semidirect product decomposition $U = HA$. Moreover, notice that A is abelian if, either $U = USp_{2n}(q)$, or $U = UO_{2n}(q)$. On the other hand, in the case where $U = UO_{2n+1}(q)$, then Φ_2^+ is a closed subset of Φ^+ and

$$A' = \prod_{\beta \in \Phi_2^+} X_\beta$$

is an abelian normal subgroup of U ; in fact, A' is the commutator subgroup $[A, A]$ of A .

We are now able to prove the following result. [Given any finite group G , we denote by $\text{Irr}(G)$ the set of all irreducible characters of G .]

Proposition 2.1. *Let $\alpha \in \Phi^+$ and $r \in \mathbb{F}_q^\times$.*

- (i) *If $U = USp_{2n}(q)$, then $\xi_{\alpha,r} \in \text{Irr}(U)$.*
- (ii) *If, either $U = UO_{2n}(q)$, or $U = UO_{2n+1}(q)$, then $\xi_{\alpha,r} \in \text{Irr}(U)$ if and only if $\alpha \in \Phi_1^+ \cup \Phi_3^+$. Moreover, if $\alpha \in \Phi_2^+$, then $\xi_{\alpha,r}$ is multiplicity free and decomposes into q irreducible constituents.*

Proof. For simplicity, we set $\lambda = \lambda_{\alpha,r}$ and $\xi = \xi_{\alpha,r}$. Let H and A be as above and let $A_\alpha = A \cap U_\alpha$. We note that $A = A_\alpha \subseteq U_\alpha$ except in the case where $U = UO_{2n+1}(q)$ and $\alpha \in \Phi_2^+$; in fact, if this is the case and $\alpha = \varepsilon_i + \varepsilon_j$ for $1 \leq i < j \leq n$, then $X_{\varepsilon_j} \not\subseteq U_\alpha$ (hence, $\varepsilon_j \notin \Phi^+(\alpha)$), whereas $X_{\varepsilon_j} \subseteq A = X_{\varepsilon_j} A_\alpha$.

Firstly, suppose that $\alpha \in \Phi_1^+$ (hence, $A_\alpha = A$). Then, we have $A \subseteq \ker(\lambda)$ and so λ may be naturally identified with a linear character $\bar{\lambda}$ of $U_\alpha/A \cong U_\alpha \cap H$. On the other hand, we also have $A \subseteq \ker(\xi)$ and so ξ may also be naturally identified with a character $\bar{\xi}$ of $U/A \cong H$. Since $\bar{\xi} = \bar{\lambda}^H$ and $H \cong U_n(q)$, we may use [3, Lemma 2] to conclude that $\bar{\xi} \in \text{Irr}(H)$. Thus, $\xi \in \text{Irr}(U)$.

Next, suppose that $\alpha \notin \Phi_1^+$. Let $\mu = \lambda_{A_\alpha}$ and let $T = I_U(\mu)$ be the inertia group of μ in U ; by the definition, T consists of all $x \in U$ satisfying $\mu(xyx^{-1}) = \mu(y)$ for all $y \in A$. It is straightforward to check that $U_\alpha \subseteq T$ and that the equality holds if and only if, either $U = USp_{2n}(q)$, or $U = UO_{2n+1}(q)$ and $\alpha \in \Phi_3^+$. In this situation, Clifford's theorem (see [13, Theorem 6.11]) guarantees that $\lambda^U \in \text{Irr}(U)$ as required.

Finally, assume that, either $U = UO_{2n}(q)$, or $U = UO_{2n+1}(q)$, and that $\alpha = \varepsilon_i + \varepsilon_j$ for some $1 \leq i < j \leq n$. Let $\beta = \varepsilon_i - \varepsilon_j$. Then, it is easy to see that $X_\beta \subseteq T$ and that, in fact, $T = X_\beta U_\alpha$; we note that $X_\beta U_\alpha$ is a subgroup of U as we easily check by using Chevalley's commutator formula (see, for example, [6, Corollary 5.2.3]). Furthermore, U_α is a normal subgroup of T and the linear character $\lambda \in \text{Irr}(U_\alpha)$ is T -invariant. Since $T = X_\beta U_\alpha$ is a semidirect product, λ extends to the linear character $\lambda': T \rightarrow \mathbb{C}^\times$ defined by $\lambda'(xz) = \lambda(z)$ for all $x \in X_\beta$ and all $z \in U_\alpha$. Therefore, by Gallagher's theorem (see [13, Corollary 6.17]), we have

$$\lambda^T = \sum_{s \in \mathbb{F}_q} \lambda' \omega_s,$$

where, for each $s \in \mathbb{F}_q$, $\omega_s \in \text{Irr}(T)$ is defined by $\omega_s(x) = \psi(sx_{i,j})$ for all $x \in T$. It follows that

$$\lambda^U = \sum_{s \in \mathbb{F}_q} (\lambda' \omega_s)^U$$

is reducible. Since the characters $\lambda' \omega_s$, for $s \in \mathbb{F}_q$, are all distinct (by Gallagher's theorem) and $T = I_U(\mu)$, Clifford's theorem implies that the characters $(\lambda' \omega_s)^U$, for $s \in \mathbb{F}_q$, are all irreducible and distinct, and this completes the proof. \square

Next, we consider the super-characters of U . Let D be a non-empty basic subset of Φ^+ and let $\phi: D \rightarrow \mathbb{F}_q^\times$ be a map. Then, as defined in the introduction, the *super-character* $\xi_{D,\phi}$ of U (associated with the pair D and ϕ) is the product of elementary characters

$$\xi_{D,\phi} = \prod_{\alpha \in D} \xi_{\alpha,\phi(\alpha)}.$$

As in the case of the unitriangular group (see [4, Theorem 1]), the super-character $\xi_{D,\phi}$ can be induced from a linear character of an appropriate subgroup of U . In fact, if we define the subgroup

$$U_D = \bigcap_{\alpha \in D} U_\alpha$$

of U and the linear character

$$\lambda_{D,\phi} = \prod_{\alpha \in D} (\lambda_{\alpha,\phi(\alpha)})_{U_D}$$

of U_D , we then have the following result.

Proposition 2.2. *Let D be a non-empty basic subset of Φ^+ and let $\phi: D \rightarrow \mathbb{F}_q^\times$ be a map. Then, $\xi_{D,\phi} = \lambda_{D,\phi}^U$.*

Proof. We proceed by induction on $|D|$, the result being obvious if $|D| = 1$. Suppose that $|D| \geq 2$, let $\alpha \in D$ be arbitrary and let $D' = D - \{\alpha\}$. By induction, we have

$$\xi_{D,\phi} = \xi_{\alpha,\phi(\alpha)} \xi_{D',\phi'} = \lambda_{\alpha,\phi(\alpha)}^U \lambda_{D',\phi'}^U,$$

where $\phi': D' \rightarrow \mathbb{F}_q^\times$ is the restriction of ϕ to D' . For simplicity, we set $\lambda = \lambda_{\alpha,\phi(\alpha)}$ and $\mu = \lambda_{D',\phi'}$. Moreover, we write $H = U_\alpha$ and $K = U_{D'}$; hence, $U_D = H \cap K$. Since D is basic, we have $HK = U$ and so $(\mu^U)_H = (\mu_{H \cap K})^H$. Thus, we deduce that

$$\xi_{D,\phi} = \lambda^U \mu^U = (\lambda(\mu^U)_H)^U = (\lambda(\mu_{H \cap K})^H)^U = ((\lambda\mu)_{H \cap K})^H)^U = \lambda_{D,\phi}^U,$$

as required. \square

3. The orbit method

In the case where the prime p is sufficiently large (in our situation, $p \geq 2n$ is enough), we may use Kirillov's method of coadjoint orbits to describe the irreducible characters of U . For a fixed positive integer n , let \mathfrak{g} denote one of the following Lie algebras defined over \mathbb{F}_q : the symplectic Lie algebra $\mathfrak{sp}_{2n}(q)$, the even orthogonal Lie algebra $\mathfrak{o}_{2n}(q)$, or the odd orthogonal Lie algebra $\mathfrak{o}_{2n+1}(q)$. Throughout the paper, we set $\mathfrak{u} = \mathfrak{g} \cap \mathfrak{u}_m(q)$ where

$$m = \begin{cases} 2n, & \text{if, either } \mathfrak{g} = \mathfrak{sp}_{2n}(q), \text{ or } \mathfrak{g} = \mathfrak{o}_{2n}(q), \\ 2n+1, & \text{if } \mathfrak{g} = \mathfrak{o}_{2n+1}(q), \end{cases}$$

and where $\mathfrak{u}_m(q)$ denotes the upper unitriangular Lie algebra consisting of all nilpotent upper-triangular $m \times m$ matrices over \mathbb{F}_q . We shall use the notation $\mathfrak{usp}_{2n}(q)$, $\mathfrak{uo}_{2n}(q)$ or $\mathfrak{uo}_{2n+1}(q)$ to refer to the Lie subalgebra \mathfrak{u} of $\mathfrak{sp}_{2n}(q)$, $\mathfrak{o}_{2n}(q)$ or $\mathfrak{o}_{2n+1}(q)$, respectively. For any $\alpha \in \Phi^+$, we will denote by e_α the matrix in \mathfrak{u} defined as follows (as usual, $1 \leq i < j \leq n$):

$$e_\alpha = \begin{cases} e_{i,j} - e_{-j,-i}, & \text{if } \alpha = \varepsilon_i - \varepsilon_j, \\ e_{i,-j} + e_{j,-i}, & \text{if } \alpha = \varepsilon_i + \varepsilon_j \text{ and } \mathfrak{u} = \mathfrak{usp}_{2n}(q), \\ e_{i,-j} - e_{j,-i}, & \text{if } \alpha = \varepsilon_i + \varepsilon_j \text{ and } \mathfrak{u} = \mathfrak{uo}_{2n}(q) \text{ or } \mathfrak{u} = \mathfrak{uo}_{2n+1}(q), \\ e_{i,-i}, & \text{if } \mathfrak{u} = \mathfrak{usp}_{2n}(q) \text{ and } \alpha = 2\varepsilon_i, \\ e_{i,0} - e_{0,-i}, & \text{if } \mathfrak{u} = \mathfrak{uo}_{2n+1}(q) \text{ and } \alpha = \varepsilon_i. \end{cases}$$

It is clear that $\{e_\alpha: \alpha \in \Phi^+\}$ is an \mathbb{F}_q -basis of \mathfrak{u} .

In the case where $p \geq 2n$, we have $u^p = 0$ for all $u \in \mathfrak{u}$ and so we may define the usual exponential map $\exp: \mathfrak{u} \rightarrow U$ by

$$\exp(u) = 1 + u + \frac{1}{2!}u^2 + \cdots + \frac{1}{n!}u^n, \quad \text{for all } u \in \mathfrak{u}.$$

It is well known that \exp is bijective and that the Campbell–Hausdorff formula holds: for all $u, v \in \mathfrak{u}$, we have

$$\exp(u)\exp(v) = \exp(u + v + \phi(u, v)),$$

where $\phi(u, v) \in [\mathfrak{u}, \mathfrak{u}]$ (see [14, p. 175]); as usual, we denote by $[\mathfrak{u}, \mathfrak{u}]$ the vector subspace of \mathfrak{u} spanned by all Lie products $[u, v]$ for $u, v \in \mathfrak{u}$. It follows that, if \mathfrak{h} is any Lie subalgebra of \mathfrak{u} , then the exponential image $H = \exp(\mathfrak{h})$ is a subgroup of U . In particular, for any $\alpha \in \Phi^+$, we have $X_\alpha = \exp(\mathbb{F}_q e_\alpha)$, and $U_\alpha = \exp(\mathfrak{u}_\alpha)$ where

$$\mathfrak{u}_\alpha = \sum_{\beta \in \Phi^+(\alpha)} \mathbb{F}_q e_\beta.$$

It follows that, for any non-empty basic subset $D \subseteq \Phi^+$, we have $U_D = \exp(\mathfrak{u}_D)$ where

$$\mathfrak{u}_D = \bigcap_{\alpha \in D} \mathfrak{u}_\alpha.$$

On the other hand, let \mathfrak{u}^* be the dual vector space of \mathfrak{u} and let $\{e_\alpha^*: \alpha \in \Phi^+\}$ be the \mathbb{F}_q -basis of \mathfrak{u}^* dual to the basis $\{e_\alpha: \alpha \in \Phi^+\}$ of \mathfrak{u} ; hence,

$$e_\alpha^*(e_\beta) = \delta_{\alpha, \beta}, \quad \text{for all } \alpha, \beta \in \Phi^+.$$

Given any $f \in \mathfrak{u}^*$, we define $B_f: \mathfrak{u} \times \mathfrak{u} \rightarrow \mathbb{F}_q$ by

$$B_f(u, v) = f([u, v]), \quad \text{for all } u, v \in \mathfrak{u}.$$

Then, B_f is a skew-symmetric bilinear form and so, if M_f denotes the matrix which represents B_f with respect to the basis $\{e_\alpha: \alpha \in \Phi^+\}$, we have $M_f^T = -M_f$. Therefore, $\text{rank } M_f$ is even and, in fact,

$$\text{rank } M_f = \dim \mathfrak{u} - \dim \text{rad}(f),$$

where

$$\text{rad}(f) = \{u \in \mathfrak{u}: f([u, v]) = 0 \text{ for all } v \in \mathfrak{u}\}$$

is the radical of B_f (see, for example, [8, Theorem 8.6.1]). By definition, a vector subspace \mathfrak{v} of \mathfrak{u} is f -isotropic (or, isotropic with respect to B_f) if $f([u, v]) = 0$ for all $u, v \in \mathfrak{v}$. By Witt's theorem (see [5, Theorems 3.10 and 3.11]), all maximal f -isotropic subspaces of \mathfrak{u} have the same dimension $\frac{1}{2}(\dim \mathfrak{u} + \dim \text{rad}(f))$. Thus, we have

$$\text{rank } M_f = 2(\dim \mathfrak{u} - \dim \mathfrak{v})$$

for any maximal f -isotropic subspace \mathfrak{v} of \mathfrak{u} .

As a first example, let $\alpha \in \Phi^+$, $r \in \mathbb{F}_q^\times$ and $f = re_\alpha^* \in \mathfrak{u}^*$. Then, it is straightforward to check that:

(i) if $\alpha = \varepsilon_i - \varepsilon_j$ for $1 \leq i < j \leq n$, then

$$\text{rank } M_f = 2(j - i - 1);$$

(ii) if $\alpha = \varepsilon_i + \varepsilon_j$ for $1 \leq i < j \leq n$, then

$$\text{rank } M_f = \begin{cases} 2(2n - i - j), & \text{if } \mathfrak{u} = \mathfrak{usp}_{2n}(q) \text{ or } \mathfrak{u} = \mathfrak{uo}_{2n+1}(q), \\ 2(2n - i - j - 1), & \text{if } \mathfrak{u} = \mathfrak{uo}_{2n}(q); \end{cases}$$

(iii) if $\alpha = 2\varepsilon_i$ for $1 \leq i \leq n$ (if $\mathfrak{u} = \mathfrak{usp}_{2n}(q)$), or $\alpha = \varepsilon_i$ for $1 \leq i \leq n$ (if $\mathfrak{u} = \mathfrak{uo}_{2n+1}(q)$), then

$$\text{rank } M_f = 2(n - i).$$

As a consequence, we deduce the following result.

Proposition 3.1. *Let $\alpha \in \Phi^+$, $r \in \mathbb{F}_q^\times$ and $f = re_\alpha^* \in \mathfrak{u}^*$.*

- (i) *If $\mathfrak{u} = \mathfrak{usp}_{2n}(q)$, then \mathfrak{u}_α is a maximal f -isotropic subspace.*
(ii) *If either $\mathfrak{u} = \mathfrak{uo}_{2n}(q)$ or $\mathfrak{u} = \mathfrak{uo}_{2n+1}(q)$, then \mathfrak{u}_α is a maximal f -isotropic subspace if and only if $\alpha \in \Phi_1^+ \cup \Phi_3^+$. Moreover, if $\alpha = \varepsilon_i + \varepsilon_j$, for $1 \leq i < j \leq n$, and if $\beta = \varepsilon_i - \varepsilon_j$, then*

$$\mathfrak{p}_\alpha = \mathfrak{u}_\alpha + \mathbb{F}_q e_\beta$$

is a maximal f -isotropic subspace of \mathfrak{u} ; in fact, for any $s \in \mathbb{F}_q$, \mathfrak{p}_α is a maximal $(f + se_\beta^)$ -isotropic subspace.*

Proof. By definition,

$$\mathfrak{u}_\alpha = \sum_{\beta \in \Phi^+(\alpha)} \mathbb{F}_q e_\beta.$$

Since $f([e_\beta, e_\gamma]) = 0$ for all $\beta, \gamma \in \Phi^+(\alpha)$, we conclude that \mathfrak{u}_α is f -isotropic. Moreover, in the case where $\alpha = \varepsilon_i + \varepsilon_j$ and, either $\mathfrak{u} = \mathfrak{uo}_{2n}(q)$, or $\mathfrak{u} = \mathfrak{uo}_{2n+1}(q)$, we clearly have $e_\beta \in \text{rad}(f)$ for $\beta = \varepsilon_i - \varepsilon_j$ and so the subspace $\mathfrak{p}_\alpha = \mathfrak{u}_\alpha + \mathbb{F}_q e_\beta$ is f -isotropic.

Now, we set

$$\Psi(\alpha) = \Phi^+ - (\Phi^+(\alpha) \cup \{\varepsilon_i - \varepsilon_j\})$$

in the case where $\alpha = \varepsilon_i + \varepsilon_j$ and, either $\mathfrak{u} = \mathfrak{uo}_{2n}(q)$, or $\mathfrak{u} = \mathfrak{uo}_{2n+1}(q)$, and

$$\Psi(\alpha) = \Phi^+ - \Phi^+(\alpha)$$

in any other case; hence, we have $\mathfrak{u} = \mathfrak{p}_\alpha \oplus \mathfrak{v}_\alpha$ where

$$\mathfrak{v}_\alpha = \sum_{\gamma \in \Psi(\alpha)} \mathbb{F}_q e_\gamma.$$

Then, for any $\gamma \in \Psi(\alpha)$, there exists a unique root $\gamma' \in \Phi^+$ such that $\gamma + \gamma' = \alpha$; hence, we have $[e_\gamma, e_{\gamma'}] = c_\gamma e_\alpha$ where $c_\gamma \in \{\pm 1, 2\}$ for $\gamma \in \Psi(\alpha)$. Let

$$\Psi'(\alpha) = \{\gamma' : \gamma \in \Psi(\alpha)\}.$$

Then, for a suitable ordering of the basis $\{e_\beta : \beta \in \Phi^+\}$, the matrix M_f can be written in the form

$$M_f = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix},$$

where X is the matrix with entries $f([e_\gamma, e_\delta])$ for $\gamma, \delta \in \Psi(\alpha) \cup \Psi'(\alpha)$. Since $f([e_\gamma, e_\delta]) = c_\gamma \delta_{\gamma', \delta}$ for all $\gamma \in \Psi(\alpha)$ and all $\delta \in \Psi(\alpha) \cup \Psi'(\alpha)$, the matrix X may be chosen with the form

$$X = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & J_t \end{pmatrix},$$

where $\Psi(\alpha) = \{\gamma_1, \dots, \gamma_t\}$ and where

$$J_k = \begin{pmatrix} 0 & c_{\gamma_k} r \\ -c_{\gamma_k} r & 0 \end{pmatrix}, \quad \text{for } 1 \leq k \leq t.$$

Since $c_\gamma \in \{\pm 1, 2\}$ for $\gamma \in \Psi(\alpha)$ and $r \neq 0$, the matrix X is non-singular and so

$$\text{rank } M_f = \text{rank } X = 2|\Psi(\alpha)| = 2 \dim \mathfrak{v}_\alpha.$$

It follows that all maximal f -isotropic subspaces of \mathfrak{u} have dimension equal to $\dim \mathfrak{u} - \dim \mathfrak{v}_\alpha = \dim \mathfrak{p}_\alpha$ and this completes the proof. \square

We observe that, in the notation of the previous proof, the subspace \mathfrak{p}_α is a Lie subalgebra of \mathfrak{u} . In general, for any $f \in \mathfrak{u}^*$, a vector subspace \mathfrak{p} of \mathfrak{u} will be called an f -polarization if \mathfrak{p} is a Lie subalgebra and also a maximal f -isotropic subspace. If this is the case (and the prime p is sufficiently large), then $P = \exp(\mathfrak{p})$ is a subgroup of U and (by the Campbell–Hausdorff formula) the map $\lambda_{f, \mathfrak{p}} : P \rightarrow \mathbb{C}^\times$, defined by the rule

$$\lambda_{f, \mathfrak{p}}(\exp(u)) = \psi(f(u)), \quad \text{for all } u \in \mathfrak{p},$$

is a linear character of P . Thus, we may consider the induced character

$$\chi_{f, \mathfrak{p}} = \lambda_{f, \mathfrak{p}}^U.$$

By [15, Lemma 1], f -polarizations always exist (see also [20, p. 114]); a detailed construction can also be found in [12, Section 1.12]. It was also proved by D. Kazhdan that the character $\chi_{f, \mathfrak{p}}$ is irreducible and does not depend on the choice of the f -polarization; hence, we may write χ_f instead of $\chi_{f, \mathfrak{p}}$. In fact, χ_f may be described in terms of the coadjoint U -orbit which contains f .

Throughout the paper, we consider the coadjoint action of U on the dual space \mathfrak{u}^* , which is defined by $(x \cdot f)(u) = f(x^{-1}ux)$ for all $x \in U$, all $f \in \mathfrak{u}^*$ and all $u \in \mathfrak{u}$, and we refer to an orbit

for this action simply as an U -orbit on \mathfrak{u}^* . Let $\mathcal{O} \subseteq \mathfrak{u}^*$ be any U -orbit and let $f \in \mathcal{O}$ be arbitrary. [For any $f \in \mathfrak{u}^*$, we will always denote by $\mathcal{O}(f)$ the coadjoint U -orbit which contains f .] Since $p \geq 2n$, the centralizer $C_U(f) = \{x \in U: x \cdot f = f\}$ is the exponential image of the radical $\text{rad}(f)$ (see [9, Lemma 1.3.1]; the proof of this lemma is easily adapted to our situation). Since $\mathcal{O} = \mathcal{O}(f)$, we conclude that

$$|\mathcal{O}| = |U| \cdot |C_U(f)|^{-1} = q^{\dim \mathfrak{u} - \dim \text{rad}(f)} = q^{\text{rank } M_f}.$$

Thus, the cardinality of any U -orbit $\mathcal{O} \subseteq \mathfrak{u}^*$ is a square power of q ; in fact, we have

$$|\mathcal{O}| = q^{2(\dim \mathfrak{u} - \dim \mathfrak{p})} = |U : \exp(\mathfrak{p})|^2$$

for any f -polarization $\mathfrak{p} \subseteq \mathfrak{u}$. Moreover, by [15, Propositions 1 and 2] (see also [20, Theorem 7.7]), the map $\chi_{\mathcal{O}} : U \rightarrow \mathbb{C}^\times$, defined by

$$\chi_{\mathcal{O}}(\exp(u)) = \frac{1}{\sqrt{|\mathcal{O}|}} \sum_{g \in \mathcal{O}} \psi(g(u)), \quad \text{for all } u \in \mathfrak{u}, \quad (3.1)$$

is an irreducible character of U , and we have $\chi_{\mathcal{O}} = \chi_f$; furthermore, the mapping $\mathcal{O} \mapsto \chi_{\mathcal{O}}$ defines a one-to-one correspondence between U -orbits on \mathfrak{u}^* and irreducible characters of U . We observe that

$$\chi_{\mathcal{O}}(1) = \sqrt{|\mathcal{O}|} = \sqrt{q^{\text{rank } M_f}} = q^{\dim \mathfrak{u} - \dim \mathfrak{p}} = |U : \exp(\mathfrak{p})|$$

for any $f \in \mathcal{O}$ and any f -polarization $\mathfrak{p} \subseteq \mathfrak{u}$.

In particular, as a consequence of Proposition 3.1, we obtain the following result (which implies Proposition 2.1). Henceforth, we will always assume that $p \geq 2n$.

Proposition 3.2. *Let $\alpha \in \Phi^+$ and $r \in \mathbb{F}_q^\times$.*

- (i) *If $U = USp_{2n}(q)$, then $\xi_{\alpha,r} = \chi_{\mathcal{O}}$ where $\mathcal{O} = \mathcal{O}(re_\alpha^*)$.*
- (ii) *Suppose that, either $U = UO_{2n}(q)$, or $U = UO_{2n+1}(q)$.*
 - (a) *If $\alpha \in \Phi_1^+ \cup \Phi_3^+$, then $\xi_{\alpha,r} = \chi_{\mathcal{O}}$ where $\mathcal{O} = \mathcal{O}(re_\alpha^*)$.*
 - (b) *If $\alpha = \varepsilon_i + \varepsilon_j$ for $1 \leq i < j \leq n$, and $\beta = \varepsilon_i - \varepsilon_j$, then*

$$\xi_{\alpha,r} = \sum_{s \in \mathbb{F}_q} \chi_s,$$

where, for any $s \in \mathbb{F}_q$, $\chi_s = \chi_{\mathcal{O}_s}$ and $\mathcal{O}_s = \mathcal{O}(re_\alpha^ + se_\beta^*)$. Moreover, we have $\langle \chi_s, \chi_{s'} \rangle = \delta_{s,s'}$ for all $s, s' \in \mathbb{F}_q$.*

Proof. The result follows immediately by Proposition 3.1 in the case where, either $U = USp_{2n}(q)$, or $\alpha \in \Phi_1^+ \cup \Phi_3^+$ and, either $U = UO_{2n}(q)$, or $U = UO_{2n+1}(q)$. In fact, in any of these situations, \mathfrak{u}_α is an (re_α^*) -polarization (by Proposition 3.1) and we may apply Kazhdan's results because $U_\alpha = \exp(\mathfrak{u}_\alpha)$ and $\lambda_{\alpha,r} = \lambda_{re_\alpha^*}$.

Now, assume that, either $U = UO_{2n}(q)$, or $U = UO_{2n+1}(q)$, and that $\alpha = \varepsilon_i + \varepsilon_j$ for $1 \leq i < j \leq n$. Let $\beta = \varepsilon_i - \varepsilon_j$, $\mathfrak{p}_\alpha = \mathfrak{u}_\alpha + \mathbb{F}_q e_\beta$ and $P_\alpha = \exp(\mathfrak{p}_\alpha)$. Moreover, let $s \in \mathbb{F}_q$ and write

$f_s = re_\alpha^* + se_\beta^* \in \mathfrak{u}^*$. Then, \mathfrak{p}_α is an f_s -polarization (by Proposition 3.1) and the linear character $\lambda_s = \lambda_{f_s, \mathfrak{p}_\alpha}$ is clearly an extension of $\lambda_{\alpha, r}$; we note that $P_\alpha = X_\beta U_\alpha$. By Kazhdan's results, we conclude that $\chi_s = \lambda_s^U$ is an irreducible constituent of $\xi_{\alpha, r} = \lambda_{\alpha, r}^U$ and the result follows by degree considerations and by Kazhdan's correspondence because, for any $s, s' \in \mathbb{F}_q$ with $s \neq s'$, the elements f_s and $f_{s'}$ are not U -conjugate. \square

For any $\alpha \in \Phi^+$ and any $r \in \mathbb{F}_q^\times$, we define $\mathcal{O}_{\alpha, r} \subseteq \mathfrak{u}^*$ as follows:

- (i) if $U = USp_{2n}(q)$, then $\mathcal{O}_{\alpha, r} = \mathcal{O}(re_\alpha^*)$;
- (ii) if, either $U = UO_{2n}(q)$, or $U = UO_{2n+1}(q)$, and $\alpha \in \Phi_1^+ \cup \Phi_3^+$, then $\mathcal{O}_{\alpha, r} = \mathcal{O}(re_\alpha^*)$;
- (iii) if, either $U = UO_{2n}(q)$, or $U = UO_{2n+1}(q)$, and $\alpha = \varepsilon_i + \varepsilon_j \in \Phi_2^+$ for $1 \leq i < j \leq n$, then

$$\mathcal{O}_{\alpha, r} = \bigcup_{s \in \mathbb{F}_q} \mathcal{O}(re_\alpha^* + se_{\varepsilon_i - \varepsilon_j}^*).$$

[An explicit description of $\mathcal{O}_{\alpha, r}$ can be found in [19]; see also [18] for the case of the orthogonal groups.]

By using Eq. (3.1), we may deduce the following result.

Proposition 3.3. *Let $\alpha \in \Phi^+$ and $r \in \mathbb{F}_q^\times$. Then,*

$$\xi_{\alpha, r}(\exp(u)) = \frac{1}{\sqrt{|\mathcal{O}(re_\alpha^*)|}} \sum_{f \in \mathcal{O}_{\alpha, r}} \psi(f(u)), \quad \text{for all } u \in \mathfrak{u}.$$

Proof. It remains to consider the case where, either $U = UO_{2n}(q)$, or $U = UO_{2n+1}(q)$, and $\alpha = \varepsilon_i + \varepsilon_j \in \Phi_2^+$ for $1 \leq i < j \leq n$. For $s \in \mathbb{F}_q$, let $\mathcal{O}_s = \mathcal{O}(re_\alpha^* + se_{\varepsilon_i - \varepsilon_j}^*)$ and let $\chi_s = \chi_{\mathcal{O}_s} \in \text{Irr}(U)$. Then, by the previous proposition (and by Eq. (3.1)), we deduce that

$$\xi_{\alpha, r}(\exp(u)) = \sum_{s \in \mathbb{F}_q} \chi_s(\exp(u)) = \sum_{s \in \mathbb{F}_q} \frac{1}{\sqrt{|\mathcal{O}_s|}} \sum_{f \in \mathcal{O}_s} \psi(f(u)).$$

The result follows because $|\mathcal{O}_s| = \chi_s(1)^2 = \chi_0(1)^2 = |\mathcal{O}(re_\alpha^*)|$ for all $s \in \mathbb{F}_q$. \square

Now, let D be a non-empty basic subset of Φ^+ , let $\phi: D \rightarrow \mathbb{F}_q^\times$ be a map and consider the super-character $\xi_{D, \phi}$ of U . We denote by $\mathcal{O}_{D, \phi}$ the subset of \mathfrak{u}^* defined by

$$\mathcal{O}_{D, \phi} = \sum_{\alpha \in D} \mathcal{O}_{\alpha, \phi(\alpha)};$$

we extend this definition to the case where D is empty by setting $\mathcal{O}_{D, \phi} = \{0\}$. We shall prove that, for any U -orbit $\mathcal{O} \subseteq \mathfrak{u}^*$, the irreducible character $\chi_{\mathcal{O}}$ is a constituent of $\xi_{D, \phi}$ if and only if $\mathcal{O} \subseteq \mathcal{O}_{D, \phi}$. Firstly, we prove the following result.

Proposition 3.4. *Let D be a basic subset of Φ^+ , let $\phi: D \rightarrow \mathbb{F}_q^\times$ be a map and define*

$$f_{D, \phi} = \sum_{\alpha \in D} \phi(\alpha) e_\alpha^* \in \mathfrak{u}_D^*;$$

if D is empty, we set $f_{D,\phi} = 0$. Let $\mathcal{O} \subseteq \mathfrak{u}^*$ be an arbitrary U -orbit. Then, $\chi_{\mathcal{O}} \in \text{Irr}(U)$ is a constituent of $\xi_{D,\phi}$ if and only if $f_{D,\phi} \in \pi(\mathcal{O})$ where $\pi: \mathfrak{u}^* \rightarrow \mathfrak{u}_D^*$ is the natural projection (sending any $f \in \mathfrak{u}^*$ to its restriction $\pi(f) = f|_{\mathfrak{u}_D}$ to \mathfrak{u}_D).

Proof. By Proposition 2.2, we know that $\xi_{D,\phi} = \lambda_{D,\phi}|_U^U$ and so

$$\langle \chi_{\mathcal{O}}, \xi_{D,\phi} \rangle_U = \langle (\chi_{\mathcal{O}})_{U_D}, \lambda_{D,\phi}|_{U_D} \rangle_{U_D}$$

(by Frobenius' reciprocity). By the definition of $\lambda_{D,\phi}$, it is clear that

$$\lambda_{D,\phi}(\exp(u)) = \psi(f_{D,\phi}(u)), \quad \text{for all } u \in \mathfrak{u}_D.$$

Since U_D centralizes $f_{D,\phi} \in \mathfrak{u}_D^*$, the coadjoint U_D -orbit

$$\mathcal{O}_{U_D}(f_{D,\phi}) = \{x \cdot f_{D,\phi} : x \in U_D\} \subseteq \mathfrak{u}_D^*$$

consists only of the element $f_{D,\phi}$ and so the result is an immediate consequence of [1, Theorem 2]. \square

On the other hand, [1, Corollary 1] implies the following result.

Proposition 3.5. *Let D be a basic subset of Φ^+ , let $\phi: D \rightarrow \mathbb{F}_q^\times$ be a map and let $\mathcal{O} \subseteq \mathfrak{u}^*$ be an arbitrary U -orbit. Then, $\chi_{\mathcal{O}} \in \text{Irr}(U)$ is a constituent of $\xi_{D,\phi}$ if and only if $\mathcal{O} \subseteq \mathcal{O}_{D,\phi}$.*

Proof. By the definition, we have

$$\xi_{D,\phi} = \prod_{\alpha \in D} \xi_{\alpha,\phi(\alpha)}.$$

Thus, in the case where $\mathfrak{u} = \mathfrak{usp}_{2n}(q)$, the result is an immediate consequence of [1, Corollary 1] because, for any $\alpha \in \Phi^+$ and any $r \in \mathbb{F}_q^\times$, the elementary character $\xi_{\alpha,r}$ is irreducible and corresponds to the U -orbit of $re_\alpha^* \in \mathfrak{u}^*$ (by Proposition 3.2). On the other hand, suppose that, either $\mathfrak{u} = \mathfrak{so}_{2n}(q)$, or $\mathfrak{u} = \mathfrak{so}_{2n+1}(q)$. Then, Proposition 3.2 implies that $\xi_{D,\phi}$ decomposes as a sum of products of the form $\prod_{\alpha \in D} \chi_\alpha$ where, for each $\alpha \in D$, $\chi_\alpha \in \text{Irr}(U)$ is a constituent of the elementary character $\xi_{\alpha,\phi(\alpha)}$ and corresponds to an U -orbit $\mathcal{O}_\alpha \subseteq \mathfrak{u}^*$ which is contained in $\mathcal{O}_{\alpha,\phi(\alpha)}$. Thus, $\chi_{\mathcal{O}}$ is a constituent of $\xi_{D,\phi}$ if and only if $\chi_{\mathcal{O}}$ is a constituent of some product $\prod_{\alpha \in D} \chi_\alpha$ and, by [1, Corollary 1], this occurs if and only if

$$\mathcal{O} \subseteq \sum_{\alpha \in D} \mathcal{O}_\alpha \subseteq \sum_{\alpha \in D} \mathcal{O}_{\alpha,\phi(\alpha)} = \mathcal{O}_{D,\phi}.$$

It follows that $\chi_{\mathcal{O}}$ is a constituent of $\xi_{D,\phi}$ if and only if $\mathcal{O} \subseteq \mathcal{O}_{D,\phi}$. \square

4. Basic subsets of u^*

In this section, we shall describe the subset $\mathcal{O}_{D,\phi} \subseteq u^*$, for any non-empty basic subset D of Φ^+ and any map $\phi: D \rightarrow \mathbb{F}_q^\times$, in terms of certain polynomial equations. Firstly, we define D -singular and D -regular entries. We define the order $<$ on \mathcal{E} as follows: for any $(i, j), (k, l) \in \mathcal{E}$, we set $(i, j) < (k, l)$ if and only if, either $l < j$, or $l = j$ and $i < k$; we recall that we are considering the mirror order $1 < 2 < \dots < n < 0 < -n < \dots < -2 < -1$. Now, for any $(i, j) \in \mathcal{E}$, we set

$$S(i, j) = \{(i, k) \in \mathcal{E}: (i, j) < (i, k)\} \cup \{(k, j) \in \mathcal{E}: (i, j) < (k, j)\},$$

and, for any $\alpha \in \Phi^+$, we define

$$\mathcal{E}_S(\alpha) = \bigcup_{(i,j) \in \mathcal{E}(\alpha)} S(i, j).$$

We say that an entry $(k, l) \in \mathcal{E}$ is α -singular if $(k, l) \in \mathcal{E}_S(\alpha)$; otherwise, we say that (k, l) is α -regular. We write $\mathcal{E}_R(\alpha)$ to denote the subset of \mathcal{E} consisting of all α -regular entries; hence,

$$\mathcal{E}_R(\alpha) = \mathcal{E} - \mathcal{E}_S(\alpha).$$

Given any basic subset $D_{\mathcal{E}}$ of \mathcal{E} , we define

$$S(D_{\mathcal{E}}) = \bigcup_{(i,j) \in D_{\mathcal{E}}} S(i, j) \quad \text{and} \quad R(D_{\mathcal{E}}) = \mathcal{E} - S(D_{\mathcal{E}}).$$

The entries in $S(D_{\mathcal{E}})$ are said to be $D_{\mathcal{E}}$ -singular, and the entries in $R(D_{\mathcal{E}})$ are said to be $D_{\mathcal{E}}$ -regular. We observe that, for $\alpha \in \Phi^+$, an entry $(k, l) \in \mathcal{E}$ is α -singular (respectively α -regular) if and only if it is $\mathcal{E}(\alpha)$ -singular (respectively $\mathcal{E}(\alpha)$ -regular) for the basic subset $\mathcal{E}(\alpha)$ of \mathcal{E} . More generally, if $D \subseteq \Phi^+$ is a basic subset, then $\mathcal{E}(D) = \bigcup_{\alpha \in D} \mathcal{E}(\alpha)$ is a basic subset of \mathcal{E} and an entry $(i, j) \in \mathcal{E}$ is called D -singular (respectively D -regular) if (i, j) is $\mathcal{E}(D)$ -singular (respectively $\mathcal{E}(D)$ -regular). We denote by $\mathcal{E}_S(D)$ the subset of \mathcal{E} consisting of all D -singular entries, and by $\mathcal{E}_R(D)$ the subset of \mathcal{E} consisting of all D -regular entries. It is clear that

$$\mathcal{E}_S(D) = \bigcup_{\alpha \in D} \mathcal{E}_S(\alpha) \quad \text{and} \quad \mathcal{E}_R(D) = \mathcal{E} - \mathcal{E}_S(D).$$

Now, let $D_{\mathcal{E}}$ be a basic subset of \mathcal{E} and let $(i, j) \in \mathcal{E}$ be arbitrary. We denote by $D_{\mathcal{E}}(i, j)$ the subset

$$D_{\mathcal{E}}(i, j) = \{(k, l) \in D_{\mathcal{E}}: 1 \preccurlyeq k < i, j < l \preccurlyeq -1\}$$

of $D_{\mathcal{E}}$; it is clear that $D_{\mathcal{E}}(i, j)$ is a basic subset of \mathcal{E} . Let

$$D_{\mathcal{E}}(i, j) = \{(i_1, j_1), \dots, (i_t, j_t)\}$$

and suppose that $j_1 < \dots < j_t$. Moreover, let $\sigma \in \text{Sym}(t)$ be the (unique) permutation such that $i_{\sigma(1)} < \dots < i_{\sigma(t)}$. [Throughout the paper, we will denote by $\text{Sym}(t)$ the symmetric group of

degree t .] Then, for any $f \in \mathfrak{u}_m(q)^*$ where, either $m = 2n$, or $m = 2n + 1$, we define $\Delta_{i,j}^{D\mathcal{E}}(f) \in \mathbb{F}_q$ to be the determinant

$$\Delta_{i,j}^{D\mathcal{E}}(f) = \begin{vmatrix} f(e_{i_{\sigma(1)},j}) & f(e_{i_{\sigma(1)},j_1}) & \cdots & f(e_{i_{\sigma(1)},j_t}) \\ \vdots & \vdots & & \vdots \\ f(e_{i_{\sigma(t)},j}) & f(e_{i_{\sigma(t)},j_1}) & \cdots & f(e_{i_{\sigma(t)},j_t}) \\ f(e_{i,j}) & f(e_{i,j_1}) & \cdots & f(e_{i,j_t}) \end{vmatrix}.$$

We note that, if $D\mathcal{E}(i, j)$ is empty, then $\Delta_{i,j}^{D\mathcal{E}}(f) = f(e_{i,j})$; in particular, if $D\mathcal{E}$ is empty, then $\Delta_{i,j}^{D\mathcal{E}}(f) = f(e_{i,j})$ for all $(i, j) \in \mathcal{E}$.

For any $f \in \mathfrak{u}^*$, we define

$$u(f) = \sum_{\alpha \in \Phi^+} u_\alpha e_\alpha \in \mathfrak{u}$$

where, for each $\alpha \in \Phi^+$, we set

$$u_\alpha = \begin{cases} \frac{1}{2} f(e_\alpha), & \text{if } \alpha \in \Phi_1^+ \cup \Phi_2^+, \\ f(e_\alpha), & \text{if } \mathfrak{u} = \mathfrak{usp}_{2n}(q) \text{ and } \alpha \in \Phi_3^+, \\ \frac{1}{2} f(e_\alpha), & \text{if } \mathfrak{u} = \mathfrak{so}_{2n+1}(q) \text{ and } \alpha \in \Phi_3^+. \end{cases}$$

It is easy to see that $f(v) = \text{tr}(u(f)^T v)$ for all $v \in \mathfrak{u}$, and that the mapping $f \mapsto u(f)$ defines a vector space isomorphism from \mathfrak{u}^* to \mathfrak{u} . Finally, we define $\hat{f} \in \mathfrak{u}_m(q)^*$ by

$$\hat{f}(v) = \text{tr}(u(f)^T v), \quad \text{for all } v \in \mathfrak{u}_m(q).$$

Then, for any basic subset D of Φ^+ and any entry $(i, j) \in \mathcal{E}$, we define $\Delta_{i,j}^D : \mathfrak{u}^* \rightarrow \mathbb{F}_q^\times$ by

$$\Delta_{i,j}^D(f) = \Delta_{i,j}^{\mathcal{E}(D)}(\hat{f}), \quad \text{for all } f \in \mathfrak{u}.$$

In particular, for any $\alpha \in \Phi^+$ and any $(i, j) \in \mathcal{E}$, we define $\Delta_{i,j}^{(\alpha)} : \mathfrak{u}^* \rightarrow \mathbb{F}_q^\times$ by

$$\Delta_{i,j}^{(\alpha)}(f) = \Delta_{i,j}^{\{\alpha\}}(f) = \Delta_{i,j}^{\mathcal{E}(\alpha)}(\hat{f}), \quad \text{for all } f \in \mathfrak{u}.$$

Given any basic subset D of Φ^+ and any map $\phi : D \rightarrow \mathbb{F}_q^\times$, we define

$$f_{D,\phi} = \sum_{\alpha \in D} \phi(\alpha) e_\alpha^* \in \mathfrak{u}^*$$

and

$$\mathcal{V}_{D,\phi} = \{f \in \mathfrak{u}^* : \Delta_{i,j}^D(f) = \Delta_{i,j}^D(f_{D,\phi}) \text{ for all } (i, j) \in \mathcal{E}_R(D)\}.$$

By the definition, we deduce that, for any $f \in \mathfrak{u}^*$, we have $f \in \mathcal{V}_{D,\phi}$ if, and only if, $\Delta_{i,j}^{\mathcal{E}(D)}(\hat{f}) = \Delta_{i,j}^{\mathcal{E}(D)}(\hat{f}_{D,\phi})$ for all $(i, j) \in \mathcal{E}_R(D)$; here, we set $\widehat{f_{D,\phi}} = \widehat{f_{D,\phi}} \in \mathfrak{u}_m(q)^*$.

On the other hand, given any basic subset $D_{\mathcal{E}} \subseteq \mathcal{E}$ and any map $\psi : D_{\mathcal{E}} \rightarrow \mathbb{F}_q^\times$, we define

$$f_{D_{\mathcal{E}}, \psi} = \sum_{(i,j) \in D_{\mathcal{E}}} \psi(i,j) e_{i,j}^* \in \mathfrak{u}_m(q)^*,$$

where $\{e_{i,j}^* : (i,j) \in \mathcal{E}\}$ is the basis of $\mathfrak{u}_m(q)^*$ dual to the standard basis $\{e_{i,j} : (i,j) \in \mathcal{E}\}$ of $\mathfrak{u}_m(q)$. Moreover, let

$$\mathcal{V}_{D_{\mathcal{E}}, \psi} = \{f \in \mathfrak{u}_m(q)^* : \Delta_{i,j}^{D_{\mathcal{E}}}(f) = \Delta_{i,j}^{D_{\mathcal{E}}}(f_{D_{\mathcal{E}}, \psi}) \text{ for all } (i,j) \in R(D_{\mathcal{E}})\}.$$

By [2, Propositions 1 and 2], the subset $\mathcal{V}_{D_{\mathcal{E}}, \psi} \subseteq \mathfrak{u}_m(q)^*$ is $U_m(q)$ -invariant where we consider the coadjoint action of $U_m(q)$ on $\mathfrak{u}_m(q)^*$; hence, $x \cdot f \in \mathcal{V}_{D_{\mathcal{E}}, \psi}$ for all $f \in \mathcal{V}_{D_{\mathcal{E}}, \psi}$ and all $x \in U_m(q)$. [In fact, we have $\mathcal{V}_{D_{\mathcal{E}}, \psi} = \mathcal{O}_{D_{\mathcal{E}}, \psi}$ where $\mathcal{O}_{D_{\mathcal{E}}, \psi} \subseteq \mathfrak{u}_m(q)^*$ is the *basic sum*

$$\mathcal{O}_{D_{\mathcal{E}}, \psi} = \sum_{(i,j) \in D_{\mathcal{E}}} \mathcal{O}(\psi(i,j) e_{i,j}^*);$$

for each $f \in \mathfrak{u}_m(q)^*$, we denote by $\mathcal{O}(f) \subseteq \mathfrak{u}_m(q)^*$ the coadjoint $U_m(q)$ -orbit which contains f .]

Finally, given any basic subset D of Φ^+ and any map $\phi : D \rightarrow \mathbb{F}_q^\times$, we define $\phi_D : \mathcal{E}(D) \rightarrow \mathbb{F}_q^\times$ by

$$\phi_D(i,j) = \hat{f}_{D, \phi}(e_{i,j}), \quad \text{for all } (i,j) \in \mathcal{E}(D);$$

hence, ϕ_D is uniquely determined by the equality

$$f_{\mathcal{E}(D), \phi_D} = \hat{f}_{D, \phi}.$$

Since $\mathcal{E}_R(D) = R(\mathcal{E}(D))$, we conclude that $\hat{f} \in \mathcal{V}_{\mathcal{E}(D), \phi_D}$ for all $f \in \mathcal{V}_{D, \phi}$, and this implies the following result.

Proposition 4.1. *Let D be a basic subset of Φ^+ and let $\phi : D \rightarrow \mathbb{F}_q^\times$ be a map. Then, $\mathcal{V}_{D, \phi}$ is U -invariant.*

Proof. Let $f \in \mathfrak{u}^*$, $u = u(f) \in \mathfrak{u}$ and $x \in U$. Since

$$(x \cdot f)(v) = f(x^{-1}vx) = \text{tr}(u^T(x^{-1}vx)) = \text{tr}((x^{-T}ux^T)^T v)$$

for all $v \in \mathfrak{u}$, we deduce that $u(x \cdot f) = (x^{-T}ux^T)^+$ where, for any $m \times m$ matrix a , we denote by $a^+ \in \mathfrak{u}_m(q)$ the upper niltriangular part of a . It follows that $x \cdot f = x \cdot \hat{f}$ and so

$$\Delta_{i,j}^D(x \cdot f) = \Delta_{i,j}^{\mathcal{E}(D)}(\widehat{x \cdot f}) = \Delta_{i,j}^{\mathcal{E}(D)}(x \cdot \hat{f}).$$

Suppose that $f \in \mathcal{V}_{D, \phi}$. Then, by the above, $\hat{f} \in \mathcal{V}_{\mathcal{E}(D), \phi_D}$ and so

$$\Delta_{i,j}^{\mathcal{E}(D)}(x \cdot \hat{f}) = \Delta_{i,j}^{\mathcal{E}(D)}(\hat{f}) = \Delta_{i,j}^D(f)$$

for all $(i, j) \in \mathcal{E}_R(D)$ (because $\mathcal{V}_{\mathcal{E}(D), \phi_D} \subseteq \mathfrak{u}_m(q)^*$ is $U_m(q)$ -invariant and $\mathcal{E}_R(D) = R(\mathcal{E}(D))$). The result follows. \square

For each $u \in \mathfrak{u}_m(q)$, let $f_u \in \mathfrak{u}_m(q)^*$ be defined by

$$f_u(v) = \text{tr}(u^T v), \quad \text{for all } v \in \mathfrak{u}_m(q);$$

thus, the mapping $u \mapsto f_u$ defines a vector space isomorphism from $\mathfrak{u}_m(q)$ to $\mathfrak{u}_m(q)^*$. For any basic subset $D_{\mathcal{E}}$ of \mathcal{E} , we clearly have

$$\Delta_{i,j}^{D_{\mathcal{E}}}(f_u) = \begin{vmatrix} u_{i_{\sigma(1)},j} & u_{i_{\sigma(1)},j_1} & \cdots & u_{i_{\sigma(1)},j_t} \\ \vdots & \vdots & & \vdots \\ u_{i_{\sigma(t)},j} & u_{i_{\sigma(t)},j_1} & \cdots & u_{i_{\sigma(t)},j_t} \\ u_{i,j} & u_{i,j_1} & \cdots & u_{i,j_t} \end{vmatrix}, \quad \text{for all } (i, j) \in \mathcal{E};$$

as in Eq. (3.1),

$$D_{\mathcal{E}}(i, j) = \{(i_1, j_1), \dots, (i_t, j_t)\},$$

where $j_1 < \cdots < j_t$, and $\sigma \in \text{Sym}(t)$ is such that $i_{\sigma(1)} < \cdots < i_{\sigma(t)}$. In the case where $u \in \mathfrak{u}$, we deduce the following easy result; given any basic subset D of Φ^+ and any $(i, j) \in \mathcal{E}$, we set

$$D(i, j) = D_{\mathcal{E}}(i, j)$$

for $D_{\mathcal{E}} = \mathcal{E}(D)$.

Lemma 4.2. *Let D be a basic subset of Φ^+ , let $u \in \mathfrak{u}$ be arbitrary and let $(i, j) \in \mathcal{E}$. Then,*

$$\Delta_{i,j}^{\mathcal{E}(D)}(f_u) = (-1)^{r+1} \Delta_{-j,-i}^{\mathcal{E}(D)}(f_u),$$

where

$$r = \begin{cases} |D(i, j)|, & \text{if, either } U = UO_{2n}(q), \text{ or } U = UO_{2n+1}(q), \\ |D(i, j) \cap \mathcal{E}(\Phi_1^+)|, & \text{if } U = USp_{2n}(q) \text{ and } (i, j) \in \mathcal{E}(\Phi_1^+), \\ -1, & \text{if } U = USp_{2n}(q) \text{ and } (i, j) \notin \mathcal{E}(\Phi_1^+). \end{cases}$$

Proof. Without loss of generality, we may assume that $1 \preccurlyeq i \preccurlyeq 0$; in fact, if $0 \prec i$, then $1 \leq -j \prec -i \prec 0$ and we may replace (i, j) by $(-j, -i)$. Let

$$D(i, j) = \{(i_1, j_1), \dots, (i_t, j_t)\},$$

where $j < j_1 < \cdots < j_t$, and let $\sigma \in \text{Sym}(t)$ be such that $i_{\sigma(1)} < \cdots < i_{\sigma(t)} < i$. By the definition of $\mathcal{E}(D)$, we clearly have

$$D(-j, -i) = \{(-j_1, -i_1), \dots, (-j_t, -i_t)\},$$

where $-j_t < \cdots < -j_1 < -j$ and $-i < -i_{\sigma(t)} < \cdots < -i_{\sigma(1)}$. Thus, we have

$$\Delta_{-j,-i}^{\mathcal{E}(D)}(f_u) = \begin{vmatrix} u_{-j_t,-i} & u_{-j_t,-i_{\sigma(t)}} & \cdots & u_{-j_t,-i_{\sigma(1)}} \\ \vdots & \vdots & & \vdots \\ u_{-j_1,-i} & u_{-j_1,-i_{\sigma(t)}} & \cdots & u_{-j_1,-i_{\sigma(1)}} \\ u_{-j,-i} & u_{-j,-i_{\sigma(t)}} & \cdots & u_{-j,-i_{\sigma(1)}} \end{vmatrix}.$$

Let $C_j, C_{j_1}, \dots, C_{j_t}$ denote the column vectors of $\Delta_{i,j}^{\mathcal{E}(D)}(f_u)$ and $L_{-j_t}, \dots, L_{-j_1}, L_{-j}$ denote the row vectors of $\Delta_{-j,-i}^{\mathcal{E}(D)}(f_u)$. Since $u \in \mathfrak{u}$, we clearly have

$$L_{-k} = \pm C_k J, \quad \text{for all } k \in \{j, j_1, \dots, j_t\},$$

where J is the $(t+1) \times (t+1)$ matrix with 1's along the anti-diagonal and 0's elsewhere. Thus, we deduce that

$$\begin{aligned} \Delta_{-j,-i}^{\mathcal{E}(D)}(f_u) &= (-1)^s \begin{vmatrix} u_{i,j_t} & u_{i_{\sigma(t)},j_t} & \cdots & u_{i_{\sigma(1)},j_t} \\ \vdots & \vdots & & \vdots \\ u_{i,j_1} & u_{i_{\sigma(t)},j_1} & \cdots & u_{i_{\sigma(1)},j_1} \\ u_{i,j} & u_{i_{\sigma(t)},j} & \cdots & u_{i_{\sigma(1)},j} \end{vmatrix} = (-1)^s \begin{vmatrix} u_{i,j_t} & \cdots & u_{i,j_1} & u_{i,j} \\ u_{i_{\sigma(t)},j_t} & \cdots & u_{i_{\sigma(t)},j_1} & u_{i_{\sigma(t)},j} \\ \vdots & & \vdots & \vdots \\ u_{i_{\sigma(1)},j_t} & \cdots & u_{i_{\sigma(1)},j_1} & u_{i_{\sigma(1)},j} \end{vmatrix} \\ &= (-1)^s \det(J)^2 \begin{vmatrix} u_{i_{\sigma(1)},j} & u_{i_{\sigma(1)},j_1} & \cdots & u_{i_{\sigma(1)},j_t} \\ \vdots & \vdots & & \vdots \\ u_{i_{\sigma(t)},j} & u_{i_{\sigma(t)},j_1} & \cdots & u_{i_{\sigma(t)},j_t} \\ u_{i,j} & u_{i,j_1} & \cdots & u_{i,j_t} \end{vmatrix} = (-1)^s \Delta_{i,j}^{\mathcal{E}(D)}(f_u), \end{aligned}$$

where

$$s = \left| \{k: k \in \{j, j_1, \dots, j_t\}, L_{-k} = -C_k J\} \right|.$$

On the one hand, assume that, either $U = UO_{2n}(q)$, or $U = UO_{2n+1}(q)$. Then, we have

$$L_{-k} = -C_k J, \quad \text{for all } k \in \{j, j_1, \dots, j_t\},$$

and so $s = t + 1$. On the other hand, suppose that $U = USp_{2n}(q)$ and let $-1 \leq r \leq t$ be such that $j_r \preccurlyeq n < j_{r+1}$; here, we set $j_{-1} = 1$ and $j_0 = j$. Then, we have

$$L_{-k} = \begin{cases} -C_k J, & \text{if } k \in \{j, j_1, \dots, j_r\}, \\ C_k J, & \text{if } k \in \{j_{r+1}, \dots, j_t\}. \end{cases}$$

Thus, $s = r + 1$ as required. \square

In particular, since $\hat{f} = f_{u(f)}$ and $u(f) \in \mathfrak{u}$, we deduce that

$$\Delta_{i,j}^D(f) = \Delta_{i,j}^{\mathcal{E}(D)}(\hat{f}) = (-1)^{r+1} \Delta_{-j,-i}^{\mathcal{E}(D)}(\hat{f}) = (-1)^{r+1} \Delta_{-j,-i}^D(f)$$

for all $(i, j) \in \mathcal{E}$ (the notation is as in the lemma). Let $\phi: D \rightarrow \mathbb{F}_q^\times$ be a map and let $f \in \mathcal{V}_{D,\phi}$ be arbitrary. Then, for any $(i, j) \in \mathcal{E}_R(D)$, we have $\Delta_{i,j}^D(f) = \Delta_{i,j}^{\mathcal{E}(D)}(\hat{f}_{D,\phi})$ and so

$$\Delta_{i,j}^D(f) = \begin{cases} (-1)^t \operatorname{sgn}(\sigma) \phi_D(i, j) \prod_{s=1}^t \phi_D(i_s, j_s), & \text{if } (i, j) \in \mathcal{E}(D), \\ 0, & \text{if } (i, j) \notin \mathcal{E}(D), \end{cases} \quad (4.1)$$

where $D(i, j) = \{(i_1, j_1), \dots, (i_t, j_t)\}$, $j_1 < \dots < j_t$, and $\sigma \in \operatorname{Sym}(t)$ is such that $i_{\sigma(1)} < \dots < i_{\sigma(t)}$.

We are now able to prove the following result.

Proposition 4.3. *Let $f \in \mathfrak{u}^*$ be arbitrary. Then, $f \in \mathcal{V}_{D,\phi}$ for some basic subset $D \subseteq \Phi^+$ and some map $\phi: D \rightarrow \mathbb{F}_q^\times$.*

Proof. Let $u = u(f) \in \mathfrak{u}$ and consider the element $f_u \in \mathfrak{u}_m(q)^*$; we note that $\hat{f} = f_u$. By [2, Proposition 3], we have $f_u \in \mathcal{V}_{D_{\mathcal{E}},\psi}$ for some basic subset $D_{\mathcal{E}}$ of \mathcal{E} and some map $\psi: D_{\mathcal{E}} \rightarrow \mathbb{F}_q^\times$. Thus, by the definition of $\mathcal{V}_{D_{\mathcal{E}},\psi}$, we have $\Delta_{i,j}^{D_{\mathcal{E}}}(f_u) = \Delta_{i,j}^{D_{\mathcal{E}}}(f_{D_{\mathcal{E}},\psi})$ for all $(i, j) \in R(D_{\mathcal{E}})$. On the other hand, let

$$v = \sum_{(i,j) \in D_{\mathcal{E}}} \psi(i, j) e_{i,j} \in \mathfrak{u}_m(q);$$

hence, $f_{D_{\mathcal{E}},\psi} = f_v$. We claim that $v \in \mathfrak{u}$. To see this, let $(i, j) \in D_{\mathcal{E}}$ be arbitrary. By Eq. (4.1), we see that

$$\Delta_{i,j}^{D_{\mathcal{E}}}(f_v) = (-1)^t \operatorname{sgn}(\sigma) \psi(i, j) \prod_{s=1}^t \psi(i_s, j_s),$$

where $D_{\mathcal{E}}(i, j) = \{(i_1, j_1), \dots, (i_t, j_t)\}$, $j_1 < \dots < j_t$, and $\sigma \in \operatorname{Sym}(t)$ is such that $i_{\sigma(1)} < \dots < i_{\sigma(t)}$. Proceeding by induction on the set \mathcal{E} (endowed with the total order \preccurlyeq), we may assume that

$$D_{\mathcal{E}}(-j, -i) = \{(-j_1, -i_1), \dots, (-j_t, -i_t)\}$$

and that the element

$$v' = \sum_{(k,l) \in D_{\mathcal{E}}(i,j) \cup D_{\mathcal{E}}(-j,-i)} \psi(k, l) e_{k,l}$$

lies in \mathfrak{u} . Then, again by Eq. (4.1), we also deduce that

$$\Delta_{-j,-i}^{D_{\mathcal{E}}}(f_v) = (-1)^t \operatorname{sgn}(\sigma) v_{-j,-i} \prod_{s=1}^t \psi(-j_s, -i_s);$$

we note that

$$v_{-j,-i} = \begin{cases} \psi(-j, -i), & \text{if } (-j, -i) \in D_{\mathcal{E}}, \\ 0, & \text{if } (-j, -i) \notin D_{\mathcal{E}}. \end{cases}$$

As in the proof of Lemma 4.2, we have

$$\prod_{s=1}^t \psi(-j_s, -i_s) = (-1)^{r'} \prod_{s=1}^t \psi(i_s, j_s),$$

where

$$r' = \begin{cases} |D(i, j)|, & \text{if, either } U = UO_{2n}(q), \text{ or } U = UO_{2n+1}(q), \\ |D(i, j) \cap \mathcal{E}(\Phi_1^+)|, & \text{if } U = USp_{2n}(q). \end{cases}$$

Thus, we obtain

$$\Delta_{-j, -i}^{D\mathcal{E}}(f_v) = (-1)^{r'} \psi(i, j)^{-1} v_{-j, -i} \Delta_{i, j}^{D\mathcal{E}}(f_v).$$

Now, since $f_u \in \mathcal{V}_{D\mathcal{E}, \psi}$ and $f_v = f_{D\mathcal{E}, \psi}$, we have

$$\Delta_{i, j}^{D\mathcal{E}}(f_u) = \Delta_{i, j}^{D\mathcal{E}}(f_v) \quad \text{and} \quad \Delta_{-j, -i}^{D\mathcal{E}}(f_u) = \Delta_{-j, -i}^{D\mathcal{E}}(f_v);$$

we note that, by induction, $(-j, -i) \in R(D\mathcal{E})$. On the other hand, since $u \in \mathfrak{u}$, we have

$$\Delta_{-j, -i}^{D\mathcal{E}}(f_u) = (-1)^{r+1} \Delta_{i, j}^{D\mathcal{E}}(f_u)$$

(by Lemma 4.2) where

$$r = \begin{cases} -1, & \text{if } U = USp_{2n}(q) \text{ and } (i, j) \notin \mathcal{E}(\Phi_1^+), \\ r', & \text{otherwise.} \end{cases}$$

Therefore, we deduce that

$$(-1)^{r'} \psi(i, j)^{-1} v_{-j, -i} \Delta_{i, j}^{D\mathcal{E}}(f_v) = (-1)^{r+1} \Delta_{i, j}^{D\mathcal{E}}(f_v)$$

and, since $\Delta_{i, j}^{D\mathcal{E}}(f_v) \neq 0$ (because $(i, j) \in D\mathcal{E}$), we obtain

$$v_{-j, -i} = \begin{cases} \psi(i, j), & \text{if } U = USp_{2n}(q) \text{ and } (i, j) \notin \mathcal{E}(\Phi_1^+), \\ -\psi(i, j), & \text{otherwise.} \end{cases}$$

It follows that $(-j, -i) \in D\mathcal{E}$ and so $v_{-j, -i} = \psi(-j, -i)$. Moreover, we conclude that $v \in \mathfrak{u}$; we observe that, in the case where, either $U = UO_{2n}(q)$, or $U = UO_{2n+1}(q)$, and $j = -i$, we conclude that $\psi(i, -i) = -\psi(i, -i)$ and this contradiction implies that $(i, -i) \notin D\mathcal{E}$. It follows that $D\mathcal{E} = \mathcal{E}(D)$ for some basic subset $D \subseteq \Phi^+$ and that $\psi = \phi_D$ for some map $\phi: D \rightarrow \mathbb{F}_q^\times$. In fact, we have $v = u(f_{D, \phi})$ and so $f_{D\mathcal{E}, \psi} = f_v = \hat{f}_{D, \phi}$. Since $\mathcal{E}_R(D) = R(\mathcal{E}(D)) = R(D\mathcal{E})$ and $\hat{f} = f_u$, we conclude that

$$\Delta_{i, j}^D(f) = \Delta_{i, j}^{\mathcal{E}(D)}(\hat{f}) = \Delta_{i, j}^{\mathcal{E}(D)}(\hat{f}_{D, \phi}) = \Delta_{i, j}^D(f_{D, \phi})$$

for all $(i, j) \in \mathcal{E}_R(D)$. Thus, $f \in \mathcal{V}_{D, \phi}$ and this completes the proof. \square

On the other hand, we also have the following proposition.

Proposition 4.4. *Let D and D' be basic subsets of Φ^+ and let $\phi: D \rightarrow \mathbb{F}_q^\times$ and $\phi': D' \rightarrow \mathbb{F}_q^\times$ be maps. Then, $\mathcal{V}_{D,\phi} \cap \mathcal{V}_{D',\phi'} \neq \emptyset$ if and only if $D = D'$ and $\phi = \phi'$.*

Proof. Suppose that $\mathcal{V}_{D,\phi} \cap \mathcal{V}_{D',\phi'} \neq \emptyset$ and let $f \in \mathcal{V}_{D,\phi} \cap \mathcal{V}_{D',\phi'}$. Then, $\hat{f} \in \mathcal{V}_{\mathcal{E}(D),\phi_D} \cap \mathcal{V}_{\mathcal{E}(D'),\phi_{D'}}$ and so, by [2, Theorem 1], we must have $\mathcal{E}(D) = \mathcal{E}(D')$ and $\phi_D = \phi_{D'}$. In particular, we deduce that $\hat{f}_{D,\phi} = \hat{f}_{D',\phi'}$ and this clearly implies that $f_{D,\phi} = f_{D',\phi'}$. Thus, we must have $D = D'$ and $\phi = \phi'$. The proof is complete. \square

Thus, we have concluded the proof of the following result.

Theorem 4.5. *The vector space \mathfrak{u}^* decomposes as the disjoint union*

$$\mathfrak{u}^* = \bigcup_{D,\phi} \mathcal{V}_{D,\phi}$$

of the U -invariant subsets $\mathcal{V}_{D,\phi}$ where D is any basic subset of Φ^+ and $\phi: D \rightarrow \mathbb{F}_q^\times$ is any map.

5. Proof of the main theorem

In this section, we prove Theorem 1.1. Firstly, we establish the equality $\mathcal{O}_{D,\phi} = \mathcal{V}_{D,\phi}$ for any basic subset $D \subseteq \Phi^+$ and any map $\phi: D \rightarrow \mathbb{F}_q^\times$. To start with, let $f \in \mathcal{O}_{D,\phi}$ be arbitrary and let $\mathcal{O} \subseteq \mathfrak{u}^*$ be the U -orbit which contains f . Then, by Proposition 3.5, we have $\langle \chi_{\mathcal{O}}, \xi_{D,\phi} \rangle \neq 0$ and so

$$\mathcal{O} \cap (f_{D,\phi} + \mathfrak{u}_D^\perp) \neq \emptyset$$

(by Proposition 3.4). Let $h \in \mathfrak{u}_D^\perp$ be such that $f_{D,\phi} + h \in \mathcal{O}$. By the definition of \mathfrak{u}_D , we have $h(e_\alpha) = 0$ for all $\alpha \in \mathcal{E}_R(D)$, and it is straightforward to check that $\hat{f}_{D,\phi} + \hat{h} \in \mathcal{V}_{\mathcal{E}(D),\phi_D}$. Therefore, by the definition of $\mathcal{V}_{D,\phi}$, we have $f_{D,\phi} + h \in \mathcal{V}_{D,\phi}$ and so $\mathcal{O} \subseteq \mathcal{V}_{D,\phi}$ (because $\mathcal{V}_{D,\phi}$ is U -invariant). It follows that $f \in \mathcal{V}_{D,\phi}$ and this completes the proof of the following result. [However, we include an alternative proof which is valid for any field (of arbitrary characteristic).]

Proposition 5.1. *If D is a basic subset of Φ^+ and $\phi: D \rightarrow \mathbb{F}_q^\times$ is a map, then $\mathcal{O}_{D,\phi} \subseteq \mathcal{V}_{D,\phi}$.*

Proof. Let $f \in \mathcal{O}_{D,\phi}$ be arbitrary. Then, by the definition of $\mathcal{O}_{D,\phi}$, we have

$$f = \sum_{\alpha \in D} f_\alpha,$$

where $f_\alpha \in \mathcal{O}_{\alpha,\phi(\alpha)}$ for all $\alpha \in D$. For each $\alpha \in D$, we define $\phi_\alpha: \mathcal{E}(\alpha) \rightarrow \mathbb{F}_q^\times$ to be the map $\phi_{\{\alpha\}}$ associated with the basic subset $\{\alpha\}$ of Φ^+ and to the restriction of ϕ to $\{\alpha\}$. Since

$$\hat{f} = \sum_{\alpha \in D} \hat{f}_\alpha \in \mathfrak{u}_m(q)^*$$

and $\hat{f}_\alpha \in \mathcal{V}_{\mathcal{E}(\alpha), \phi_\alpha}$ for all $\alpha \in D$, we conclude that

$$\hat{f} \in \sum_{\alpha \in D} \mathcal{V}_{\mathcal{E}(\alpha), \phi_\alpha}.$$

Since $\mathcal{V}_{\mathcal{E}(\alpha), \phi_\alpha} = \mathcal{O}_{\mathcal{E}(\alpha), \phi_\alpha}$ (by [2, Proposition 2]), we deduce that

$$\hat{f} \in \sum_{\alpha \in D} \mathcal{O}_{\mathcal{E}(\alpha), \phi_\alpha} = \mathcal{O}_{\mathcal{E}(D), \phi_D}.$$

Since $\mathcal{O}_{\mathcal{E}(D), \phi_D} = \mathcal{V}_{\mathcal{E}(D), \phi_D}$ (by [2, Proposition 2]), it follows that $\hat{f} \in \mathcal{V}_{\mathcal{E}(D), \phi_D}$ and this implies that $f \in \mathcal{V}_{D, \phi}$ (by the definition of $\mathcal{V}_{D, \phi}$). The proof is complete. \square

The reverse inclusion will be proved by induction on the cardinality of the basic subset D . We start by proving the following lemma.

Lemma 5.2. *Let D be a non-empty basic subset of Φ^+ and let $\phi: D \rightarrow \mathbb{F}_q^\times$ be a map. Let $(i, j) \in \mathcal{E}(D)$ be the smallest entry of $\mathcal{E}(D)$ (with respect to the order \preceq) and let $\alpha \in D$ be such that $(i, j) \in \mathcal{E}(\alpha)$. Let $D' = D - \{\alpha\}$ and let $\phi': D' \rightarrow \mathbb{F}_q^\times$ be restriction of ϕ to D' . Then, $\mathcal{V}_{D, \phi} = \mathcal{O}_{\alpha, \phi(\alpha)} + \mathcal{V}_{D', \phi'}$.*

Proof. We first note that, by [2, Proposition 2],

$$\mathcal{V}_{\mathcal{E}(D), \phi_D} = \mathcal{O}_{\mathcal{E}(D), \phi_D} = \mathcal{O}_{\mathcal{E}(\alpha), \phi_\alpha} + \mathcal{O}_{\mathcal{E}(D'), \phi_{D'}} = \mathcal{O}_{\mathcal{E}(\alpha), \phi_\alpha} + \mathcal{V}_{\mathcal{E}(D'), \phi_{D'}},$$

where ϕ_α is as in the previous proof and where $\phi_{D'}$ denotes the restriction of ϕ_D to $\mathcal{E}(D')$. Let $f \in \mathcal{O}_{\alpha, \phi(\alpha)} + \mathcal{V}_{D', \phi'}$ be arbitrary. Then,

$$\hat{f} \in \mathcal{O}_{\mathcal{E}(\alpha), \phi_\alpha} + \mathcal{V}_{\mathcal{E}(D'), \phi_{D'}} = \mathcal{V}_{\mathcal{E}(D), \phi_D}$$

and so $f \in \mathcal{V}_{D, \phi}$. On the other hand, let $f \in \mathcal{V}_{D, \phi}$ be arbitrary. Moreover, let $S(\alpha) \subseteq \Phi^+$ be the set consisting of all roots $\beta \in \Phi^+$ for which there exists $\gamma \in \Phi^+$ such that $\beta + \gamma = \alpha$. Then,

$$S(\alpha) \subseteq \{\beta \in \Phi^+ : \mathcal{E}(\beta) \subseteq \mathcal{E}_S(\alpha)\}$$

and, in fact, the equality holds in all cases except if, either $u = u\sigma_{2n}(q)$, or $u = u\sigma_{2n+1}(q)$, and $\alpha \in \Phi_2^+$; if this is the case, we must have $\alpha = \varepsilon_{-j} + \varepsilon_i$ and the root $\beta = \varepsilon_{-j} - \varepsilon_i$ satisfies $\mathcal{E}(\beta) \subseteq \mathcal{E}_S(\alpha)$ but does not lie in $S(\alpha)$. It is not difficult to show that there exists $x \in U$ such that the element $x \cdot f \in u^*$ satisfies $(x \cdot f)(e_\beta) = 0$ for all $\beta \in S(\alpha)$. This implies that $x \cdot f = f_\alpha + g$ where $f_\alpha \in \mathcal{O}_{\alpha, \phi(\alpha)}$ satisfies $f_\alpha(e_\beta) = 0$ for all $\beta \in S(\alpha)$ and where $g \in u^*$ satisfies $g(e_\beta) = 0$ for all $\beta \in \Phi^+$ with $\mathcal{E}(\beta) \subseteq \mathcal{E}_S(\alpha)$. Now, since $\mathcal{V}_{D, \phi}$ is U -invariant, we have $x \cdot f \in \mathcal{V}_{D, \phi}$ and it is straightforward to check that the equations which define $\mathcal{V}_{D, \phi}$ imply that $g \in \mathcal{V}_{D', \phi'}$. It follows that $f \in \mathcal{O}_{\alpha, \phi(\alpha)} + \mathcal{V}_{D', \phi'}$ and this completes the proof. \square

As a consequence, we obtain the following result (see [19] for a more direct proof).

Corollary 5.3. *If $\alpha \in \Phi^+$ and $r \in \mathbb{F}_q^\times$, then $\mathcal{V}_{\alpha, r} = \mathcal{O}_{\alpha, r}$.*

Proof. Let $D = \{\alpha\}$ and let $\phi: D \rightarrow \mathbb{F}_q^\times$ be defined $\phi(\alpha) = r$. Then, in the notation of the lemma, we have $\mathcal{V}_{\alpha,r} = \mathcal{V}_{D,\phi} = \mathcal{O}_{\alpha,\phi(\alpha)} + \mathcal{V}_{D',\phi'}$. Since D' is empty, we have $\mathcal{V}_{D',\phi'} = \{0\}$ and so $\mathcal{V}_{\alpha,r} = \mathcal{O}_{\alpha,\phi(\alpha)}$ as required. \square

Finally, we prove the following result.

Proposition 5.4. *If D is a basic subset of Φ^+ and $\phi: D \rightarrow \mathbb{F}_q^\times$ is a map, then $\mathcal{V}_{D,\phi} \subseteq \mathcal{O}_{D,\phi}$.*

Proof. We proceed by induction on $|D|$. The case $|D| = 1$ follows by the previous corollary. Thus, suppose that $|D| \geq 2$ and let $\alpha \in D$, $D' = D - \{\alpha\}$ and $\phi': D' \rightarrow \mathbb{F}_q^\times$ be as in Lemma 5.2. Then, by that lemma, we have

$$\mathcal{V}_{D,\phi} = \mathcal{O}_{\alpha,\phi(\alpha)} + \mathcal{V}_{D',\phi'}.$$

By induction, we conclude that $\mathcal{V}_{D',\phi'} \subseteq \mathcal{O}_{D',\phi'}$ and so

$$\mathcal{V}_{D,\phi} \subseteq \mathcal{O}_{\alpha,\phi(\alpha)} + \mathcal{O}_{D',\phi'} = \mathcal{O}_{D,\phi}.$$

The result follows. \square

We have therefore concluded the proof of the following result.

Theorem 5.5. *If D is a basic subset of Φ^+ and $\phi: D \rightarrow \mathbb{F}_q^\times$ is a map, then $\mathcal{V}_{D,\phi} = \mathcal{O}_{D,\phi}$.*

We are now able to proceed with the proof of our main result.

Proof of Theorem 1.1. Firstly, suppose that $\langle \xi_{D,\phi}, \xi_{D',\phi'} \rangle \neq 0$ for some basic subsets $D, D' \subseteq \Phi^+$ and some maps $\phi: D \rightarrow \mathbb{F}_q^\times$ and $\phi': D' \rightarrow \mathbb{F}_q^\times$. Then, there exists $\chi \in \text{Irr}(U)$ with $\langle \chi, \xi_{D,\phi} \rangle \neq 0$ and $\langle \chi, \xi_{D',\phi'} \rangle \neq 0$. By Kazhdan's correspondence, there exists a (unique) U -orbit $\mathcal{O} \subseteq \mathfrak{u}^*$ such that $\chi = \chi_{\mathcal{O}}$. Then, by Proposition 3.5, we have $\mathcal{O} \subseteq \mathcal{O}_{D,\phi} \cap \mathcal{O}_{D',\phi'}$ and so $\mathcal{O} \subseteq \mathcal{V}_{D,\phi} \cap \mathcal{V}_{D',\phi'}$ (by Theorem 5.5). By Theorem 4.5, we conclude that $D = D'$ and $\phi = \phi'$.

On the other hand, let $\chi \in \text{Irr}(U)$ be arbitrary, let $\mathcal{O} \subseteq \mathfrak{u}^*$ be the (unique) U -orbit with $\chi = \chi_{\mathcal{O}}$ and let $f \in \mathcal{O}$. By Theorem 4.5, we know that $f \in \mathcal{V}_{D,\phi}$ for some basic subset $D \subseteq \Phi^+$ and some map $\phi: D \rightarrow \mathbb{F}_q^\times$. Therefore, $f \in \mathcal{O}_{D,\phi}$ (by Theorem 5.5) and so $\langle \chi, \xi_{D,\phi} \rangle \neq 0$ (by Proposition 3.5).

The proof is complete. \square

6. Two examples

In this section, we illustrate our results with two examples. Firstly, we consider the linear characters of U and, on the other extreme, we determine all the irreducible characters with maximum degree. [Here, as in the previous sections, we continue to assume that $p \geq 2n$.]

6.1. Linear characters

Let Π be the subset of Φ^+ consisting of all *simple roots*. By definition, a root $\alpha \in \Phi^+$ is said to be *simple* if it cannot be expressed as a sum of two other (positive) roots. Hence, we have

$$\Pi = \begin{cases} \{\varepsilon_i - \varepsilon_{i+1} : 1 \leq i < n\} \cup \{2\varepsilon_n\}, & \text{if } G = Sp_{2n}(q), \\ \{\varepsilon_i - \varepsilon_{i+1} : 1 \leq i < n\} \cup \{\varepsilon_{n-1} + \varepsilon_n\}, & \text{if } G = O_{2n}(q), \\ \{\varepsilon_i - \varepsilon_{i+1} : 1 \leq i < n\} \cup \{\varepsilon_n\}, & \text{if } G = O_{2n+1}(q). \end{cases}$$

We observe that, by Chevalley's commutator formula (see [6, Corollary 5.2.3]), we have

$$U' = \prod_{\alpha \in \Phi^+ - \Pi} X_\alpha,$$

where $U' = [U, U]$ is the commutator subgroup of U . Thus, we have $|U : U'| = q^{|\Pi|} = q^n$ and so U has exactly q^n linear characters.

Now, suppose that, either $U = USp_{2n}(q)$, or $U = UO_{2n+1}(q)$. Then, it is clear that any subset of Π is a basic subset of Φ^+ and that, for any $D \subseteq \Pi$ and any map $\phi : D \rightarrow \mathbb{F}_q^\times$, the super-character $\xi_{D,\phi}$ is linear (because $U_D = U$). By Theorem 1.1, we conclude that U has at least

$$\sum_{k=0}^n \binom{n}{k} (q-1)^k = q^n$$

linear characters and this completes the proof of the following result.

Proposition 6.1. *Suppose that, either $U = USp_{2n}(q)$, or $U = UO_{2n+1}(q)$, and let $\vartheta \in \text{Irr}(U)$. Then, ϑ is linear if and only if $\vartheta = \xi_{D,\phi}$ for some $D \subseteq \Pi$ and some map $\phi : D \rightarrow \mathbb{F}_q^\times$.*

On the other hand, suppose that $U = UO_{2n}(q)$. Then, a subset D of Π is a basic subset of Φ^+ if and only if D contains at most one of the roots $\varepsilon_{n-1} - \varepsilon_n$ or $\varepsilon_{n-1} + \varepsilon_n$. Moreover, if $D \subset \Pi$ is basic and $\varepsilon_{n-1} + \varepsilon_n \notin D$, then $U_D = U$ and, for any map $\phi : D \rightarrow \mathbb{F}_q^\times$, the super-character $\xi_{D,\phi}$ is clearly linear. Otherwise, suppose that D is a basic subset of Π with $\varepsilon_{n-1} + \varepsilon_n \in D$, let $\phi : D \rightarrow \mathbb{F}_q^\times$ be any map and write $r = \phi(\varepsilon_{n-1} + \varepsilon_n)$. By Proposition 2.1 (or by Proposition 3.1), the elementary character $\xi_{\alpha,r}$ is multiplicity free and decomposes as a sum of q linear characters. Let μ_1, \dots, μ_q be the linear constituents of $\xi_{\alpha,r}$, let $D' = D - \{\varepsilon_{n-1} + \varepsilon_n\}$ and let $\phi' : D' \rightarrow \mathbb{F}_q^\times$ be the restriction of ϕ to ϕ' . Then, we deduce that

$$\xi_{D,\phi} = \mu_1 \xi_{D',\phi'} + \dots + \mu_q \xi_{D',\phi'}$$

also decomposes as a sum of q distinct linear characters. By Theorem 1.1, we conclude that U has at least

$$\sum_{k=0}^{n-1} \binom{n-1}{k} (q-1)^k + q \sum_{k=0}^{n-2} \binom{n-2}{k} (q-1)^{k+1} = q^{n-1} + q(q-1)q^{n-2} = q^n$$

linear characters. As above, we have completed the proof of the following result.

Proposition 6.2. Suppose that $U = UO_{2n}(q)$ and let $\vartheta \in \text{Irr}(U)$. Then, ϑ is linear if and only if one of the following cases occurs.

- (i) $\vartheta = \xi_{D,\phi}$ for some $D \subseteq \Pi - \{\varepsilon_{n-1} + \varepsilon_n\}$ and some map $\phi: D \rightarrow \mathbb{F}_q^\times$.
- (ii) ϑ is a constituent of $\xi_{\varepsilon_{n-1} + \varepsilon_n, r} \xi_{D,\phi}$ for some $D \subseteq \Pi - \{\varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}$, some map $\phi: D \rightarrow \mathbb{F}_q^\times$ and some $r \in \mathbb{F}_q^\times$.

6.2. Irreducible characters of maximum degree: the symplectic case

In this subsection, we will always assume that $U = USp_{2n}(q)$ and determine all the irreducible characters of maximum degree of U . As in Section 2,

$$A = \prod_{\alpha \in \Phi_2^+ \cup \Phi_3^+} X_\alpha$$

is an abelian normal subgroup of U and so, by Ito's Theorem (see [13, Theorem 6.15]), every irreducible character of U has degree less or equal to $|U : A| = q^{n(n-1)/2}$. We shall prove $q^{n(n-1)/2}$ is, in fact, the maximum degree of the irreducible characters of U and that every irreducible character of U with maximum degree is a super-character corresponding to a certain maximal basic subset which is contained in the set

$$\Gamma = \Phi_3^+ \cup \{\varepsilon_i + \varepsilon_{i+1} : 1 \leq i < n\}.$$

We start by proving the following result.

Proposition 6.3. Let D be a basic subset of Γ and $\phi: D \rightarrow \mathbb{F}_q^\times$ be a map. Then, the super-character $\xi_{D,\phi}$ is irreducible. In particular, in the case where, either D , or $D \cup \{2\varepsilon_n\}$, is a maximal basic subset of Γ , the super-character $\xi_{D,\phi}$ is irreducible and has maximum degree $\xi_{D,\phi}(1) = q^{n(n-1)/2}$.

Proof. Let $\alpha \in D$ be such that $\mathcal{E}(\alpha) = \{(i, -j), (j, -i)\}$ and $l < -j$ for all $(k, l) \in \mathcal{E}(D)$; we note that, either $j = i + 1$, or $j = i$. Let $D' = D - \{\alpha\}$ and let $\phi': D' \rightarrow \mathbb{F}_q^\times$ be the restriction of ϕ to D' . Moreover, let $-\phi': D' \rightarrow \mathbb{F}_q^\times$ be defined by $(-\phi')(\beta) = -\phi'(\beta) = -\phi(\beta)$ for all $\beta \in D'$. It is easy to check that

$$\overline{\xi_{D',\phi'}} = \xi_{D',-\phi'}$$

where, for any character χ of U , we denote by $\bar{\chi}$ the character defined by $\bar{\chi}(x) = \overline{\chi(x)} = \chi(x^{-1})$ for all $x \in U$. Since $\xi_{D,\phi} = \xi_{\alpha,\phi(\alpha)} \xi_{D',\phi'}$, we deduce that

$$\langle \xi_{D,\phi}, \xi_{D,\phi} \rangle = \langle \xi_{\alpha,\phi(\alpha)}, \xi_{\alpha,\phi(\alpha)} \xi_{D',\phi'} \overline{\xi_{D',\phi'}} \rangle = \langle \xi_{\alpha,\phi(\alpha)}, \xi_{\alpha,\phi(\alpha)} \xi_{D',\phi'} \xi_{D',-\phi'} \rangle.$$

Let $\chi \in \text{Irr}(U)$ be any irreducible constituent of $\xi_{D',\phi'} \xi_{D',-\phi'}$. By Theorem 1.1, there exists a unique basic subset D_1 of Φ^+ and a unique map $\phi_1: D_1 \rightarrow \mathbb{F}_q^\times$ such that $\langle \chi, \xi_{D_1,\phi_1} \rangle \neq 0$. On the other hand, by Kazhdan's correspondence, we know that $\chi = \chi_{\mathcal{O}}$ for a unique U -orbit $\mathcal{O} \subseteq \mathfrak{u}^*$. Furthermore, by [1, Corollary 1], we have

$$\mathcal{O} \subseteq \mathcal{O}_{D_1,\phi_1} \cap (\mathcal{O}_{D',\phi'} + \mathcal{O}_{D',-\phi'}).$$

Since $\mathcal{O}_{D_1, \phi_1} = \mathcal{V}_{D_1, \phi_1}$, it is clear that

$$\mathcal{E}(D_1) \subseteq \{(k, l) \in \mathcal{E}(\Phi^+): j < k < l < -j\}.$$

Thus, $D_1 \cup \{\alpha\}$ is a basic subset of Φ^+ and so $\langle \xi_{\alpha, \phi(\alpha)}, \xi_{\alpha, \phi(\alpha)} \xi_{D_1, \phi_1} \rangle \neq 0$ if and only if D_1 is empty (and $\xi_{D_1, \phi_1} = 1_U$). It follows that

$$\langle \xi_{\alpha, \phi(\alpha)}, \xi_{\alpha, \phi(\alpha)} \xi_{D', \phi'} \xi_{D', -\phi'} \rangle = \langle \xi_{\alpha, \phi(\alpha)}, \xi_{\alpha, \phi(\alpha)} \rangle \cdot \langle \xi_{D', \phi'} \xi_{D', -\phi'}, 1_U \rangle.$$

Since $\xi_{\alpha, \phi(\alpha)} \in \text{Irr}(U)$ and $\xi_{D', -\phi'} = \overline{\xi_{D', \phi'}}$, we conclude that

$$\langle \xi_{D, \phi}, \xi_{D, \phi} \rangle = \langle \xi_{D', \phi'}, \xi_{D', \phi'} \rangle$$

and this clearly implies that $\langle \xi_{D, \phi}, \xi_{D, \phi} \rangle = 1$. The result follows. \square

Now, let $\chi \in \text{Irr}(U)$ and suppose that $\chi(1) = q^{n(n-1)/2}$. Let D be the unique basic subset of Φ^+ and $\phi: D \rightarrow \mathbb{F}_q^\times$ the unique map such that $\langle \chi, \xi_{D, \phi} \rangle \neq 0$. Since $\xi_{D, \phi}(1) = |U : U_D|$ (by Proposition 2.2) and $A \subseteq U_D$, we have $|U : U_D| \leq q^{n(n-1)/2}$ and so $\xi_{D, \phi}(1) = \chi(1) = q^{n(n-1)/2}$. Thus, $\chi = \xi_{D, \phi}$ and, moreover, $U_D = A$. By the definition of U_D , we conclude that, either D , or $D \cup \{2\varepsilon_n\}$, is a maximal basic subset of $\Phi_2^+ \cup \Phi_3^+$. We claim that $D \subseteq \Gamma$. To see this, suppose that $D \not\subseteq \Gamma$ and let $\alpha \in D - \Gamma$. Then, $\alpha = \varepsilon_i + \varepsilon_j$ for some $1 \leq i < j \leq n$ with $j \neq i + 1$. Moreover, by the maximality of D , there should exist $i + 1 \leq k \leq n$ such that $\beta = \varepsilon_{i+1} + \varepsilon_k \in D$; we note that $\beta = 2\varepsilon_{i+1} \in \Phi_3^+$ in the case where $k = i + 1$. We shall now prove that the super-character $\xi_{D, \phi}$ is reducible. Otherwise, by Kazhdan's correspondence and by [1, Corollary 1], the sum $\mathcal{O}_{D, \phi}$ must be the coadjoint U -orbit which corresponds to the irreducible character $\xi_{D, \phi}$. In particular, we must have $|\mathcal{O}_{D, \phi}| = \xi_{D, \phi}(1)^2 = q^{n(n-1)}$ and, since

$$f_{D, \phi} = \sum_{\gamma \in D} \phi(\gamma) e_\gamma^* \in \mathcal{O}_{D, \phi},$$

we easily conclude that

$$\mathfrak{a} = \sum_{\gamma \in \Phi_2^+ \cup \Phi_3^+} \mathbb{F}_q e_\gamma$$

is an $f_{D, \phi}$ -polarization. However, it is straightforward to check that the vector

$$u = \begin{cases} e_{\varepsilon_{i+1} - \varepsilon_j} - \phi(\alpha)^{-1} \phi(\beta) e_{\varepsilon_i - \varepsilon_k}, & \text{if } j < k, \\ e_{\varepsilon_i - \varepsilon_{i+1}} - \phi(\beta)^{-1} \phi(\alpha) e_{\varepsilon_k - \varepsilon_j}, & \text{if } k < j, \end{cases}$$

lies in $\text{rad}(f_{D, \phi})$ and so $\mathfrak{a} + \mathbb{F}_q u$ is an $f_{D, \phi}$ -isotropic subspace of u , a contradiction (because $u \notin \mathfrak{a}$). Thus, $\xi_{D, \phi}$ is reducible and so

$$\chi(1) < \xi_{D, \phi}(1) \leq q^{n(n-1)/2},$$

another contradiction. It follows that $D \subseteq \Gamma$ and this completes the proof of the following result.

Theorem 6.4. *Let $\chi \in \text{Irr}(U)$. Then, χ has maximum degree if and only if $\chi = \xi_{D,\phi}$ where, either D , or $D \cup \{2\varepsilon_n\}$, is a maximal basic subset of Γ and where $\phi: D \rightarrow \mathbb{F}_q^\times$ is any map. Furthermore, if this is the case, we have $\chi(1) = \xi_{D,\phi}(1) = q^{n(n-1)/2}$.*

6.3. Irreducible characters of maximum degree: the orthogonal case

Finally, we consider the case where, either $U = UO_{2n}(q)$, or $U = UO_{2n+1}(q)$. In either case,

$$A = \prod_{\alpha \in \Phi_2^+} X_\alpha$$

is an abelian normal subgroup of U and so, by Ito's Theorem, every irreducible character has degree less or equal to

$$|U : A| = \begin{cases} q^{n(n-1)/2}, & \text{if } U = UO_{2n}(q), \\ q^{n(n+1)/2}, & \text{if } U = UO_{2n+1}(q). \end{cases}$$

In the present situation, we shall prove that every irreducible character of U with maximum degree occurs as a constituent of a super-character corresponding to a certain maximal basic subset which is almost entirely contained in the set

$$\Gamma = \{\varepsilon_i + \varepsilon_{i+1} : 1 \leq i < n\}.$$

As in the symplectic case, we start by proving the following result.

Proposition 6.5. *Let D be a basic subset of Γ and let $\phi: D \rightarrow \mathbb{F}_q^\times$ be a map. Then, the super-character $\xi_{D,\phi}$ is multiplicity free and has $q^{|D|}$ irreducible constituents, each with degree equal to $q^{-|D|}\xi_{D,\phi}(1)$.*

Proof. Let $\alpha = \varepsilon_i + \varepsilon_{i+1} \in D$, $1 \leq i < n$, be such that $i < k$ whenever $\varepsilon_k + \varepsilon_{k+1} \in D$. Let $D' = D - \{\alpha\}$ and let $\phi': D' \rightarrow \mathbb{F}_q^\times$ be the restriction of ϕ to D' . Moreover, let $-\phi': D' \rightarrow \mathbb{F}_q^\times$ be defined by $(-\phi')(\beta) = -\phi'(\beta) = -\phi(\beta)$ for all $\beta \in D'$. Then, we have $\overline{\xi_{D',\phi'}} = \xi_{D',-\phi'}$ and, as in the proof of Proposition 6.3, we deduce that

$$\begin{aligned} \langle \xi_{D,\phi}, \xi_{D,\phi} \rangle &= \langle \xi_{\alpha,\phi(\alpha)}, \xi_{\alpha,\phi(\alpha)} \xi_{D',\phi'} \xi_{D',-\phi'} \rangle \\ &= \langle \xi_{\alpha,\phi(\alpha)}, \xi_{\alpha,\phi(\alpha)} \rangle \cdot \langle \xi_{D',\phi'} \xi_{D',-\phi'}, 1_U \rangle \\ &= \langle \xi_{\alpha,\phi(\alpha)}, \xi_{\alpha,\phi(\alpha)} \rangle \cdot \langle \xi_{D',\phi'}, \xi_{D',\phi'} \rangle. \end{aligned}$$

By Proposition 2.1 (or by Proposition 3.1), we know that $\xi_{\alpha,\phi(\alpha)}$ is multiplicity free and has q irreducible constituents τ_1, \dots, τ_q , each with degree $q^{-1}\xi_{\alpha,\phi(\alpha)}(1)$. Thus, we conclude that

$$\langle \xi_{D,\phi}, \xi_{D,\phi} \rangle = q \langle \xi_{D',\phi'}, \xi_{D',\phi'} \rangle.$$

Let us assume (by induction) that $\xi_{D',\phi'}$ is multiplicity free and has $t = q^{|D'|}$ irreducible constituents $\sigma_1, \dots, \sigma_t$, each with degree $q^{-|D'|}\xi_{D',\phi'}(1)$. In particular, we have $\langle \xi_{D',\phi'}, \xi_{D',\phi'} \rangle = q^{|D'|}$ and so

$$\langle \xi_{D,\phi}, \xi_{D,\phi} \rangle = q^{|D'|+1} = q^{|D|}.$$

Since

$$\xi_{D,\phi} = \xi_{\alpha,\phi(\alpha)} \xi_{D',\phi'} = \sum_{1 \leq r \leq q} \sum_{1 \leq s \leq t} \tau_r \sigma_s,$$

we obtain

$$q^{|D|} = \langle \xi_{D,\phi}, \xi_{D,\phi} \rangle = \sum_{1 \leq r, r' \leq q} \sum_{1 \leq s, s' \leq t} \langle \tau_r \sigma_s, \tau_{r'} \sigma_{s'} \rangle$$

and this clearly implies that

$$\langle \tau_r \sigma_s, \tau_{r'} \sigma_{s'} \rangle = \delta_{r,r'} \delta_{s,s'}, \quad 1 \leq r, r' \leq q, \quad 1 \leq s, s' \leq t.$$

The result follows. \square

In particular, let

$$D = \{\varepsilon_1 + \varepsilon_2, \varepsilon_3 + \varepsilon_4, \dots, \varepsilon_{2r-1} + \varepsilon_{2r}\},$$

where, either $n = 2r$, or $n = 2r + 1$; hence, D is a maximal basic subset of Γ (however, notice that not every maximal basic subset of Γ has this form). Then, for any map $\phi: D \rightarrow \mathbb{F}_q^\times$, the super-character $\xi_{D,\phi}$ has q^r (distinct) irreducible constituents, each with degree equal to $q^{d(n)}$ where

$$d(n) = \begin{cases} n(n-2)/2, & \text{if } U = UO_{2n}(q) \text{ and } n \text{ is even,} \\ (n-1)^2/2, & \text{if } U = UO_{2n}(q) \text{ and } n \text{ is odd,} \\ n(n-1)/2, & \text{if } U = UO_{2n+1}(q). \end{cases}$$

Furthermore, we observe that, in the case where $U = UO_{2n}(q)$ and $n = 2r$,

$$D = \{\varepsilon_1 + \varepsilon_2, \varepsilon_3 + \varepsilon_4, \dots, \varepsilon_{2r-3} + \varepsilon_{2r-2}\}$$

is a basic subset of Γ and so, for any map $\phi: D \rightarrow \mathbb{F}_q^\times$, the super-character $\xi_{D,\phi}$ has q^{r-1} (distinct) irreducible constituents, each with degree equal to $q^{n(n-2)/2}$. Hence, for

$$D = \{\varepsilon_1 + \varepsilon_2, \varepsilon_3 + \varepsilon_4, \dots, \varepsilon_{2r-3} + \varepsilon_{2r-2}\} \cup \{\varepsilon_{2r-1} - \varepsilon_{2r}\}$$

and for any map $\phi: D \rightarrow \mathbb{F}_q^\times$, the super-character $\xi_{D,\phi}$ also has q^{r-1} (distinct) irreducible constituents, each with degree equal to $q^{n(n-2)/2}$. On the other hand, in the case where $U = UO_{2n+1}(q)$, let

$$D = \begin{cases} \{\varepsilon_1 + \varepsilon_2, \varepsilon_3 + \varepsilon_4, \dots, \varepsilon_{2r-3} + \varepsilon_{2r-2}\} \cup \{\varepsilon_{2r-1}\}, & \text{if } n = 2r, \\ \{\varepsilon_1 + \varepsilon_2, \varepsilon_3 + \varepsilon_4, \dots, \varepsilon_{2r-1} + \varepsilon_{2r}\} \cup \{\varepsilon_{2r+1}\}, & \text{if } n = 2r + 1, \end{cases}$$

and let $\phi: D \rightarrow \mathbb{F}_q^\times$ be any map. Then, the super-character $\xi_{D,\phi}$ also has (either q^{r-1} , or q^r) irreducible constituents, each with degree equal to $q^{n(n-1)/2}$. In fact, in the case where $n = 2r + 1$, it is enough to observe that the character $\xi_{\varepsilon_{2r+1}, \phi(\varepsilon_{2r+1})}$ is linear. Otherwise, suppose that $n = 2r$ and let

$$D' = \{\varepsilon_1 + \varepsilon_2, \varepsilon_3 + \varepsilon_4, \dots, \varepsilon_{2r-3} + \varepsilon_{2r-2}\}.$$

Then, we have

$$\langle \xi_{D,\phi}, \xi_{D,\phi} \rangle = \langle \xi_{D',\phi'}, \xi_{D',\phi'} \xi_{\alpha, \phi(\alpha)} \xi_{\alpha, -\phi(\alpha)} \rangle,$$

where $\phi': D' \rightarrow \mathbb{F}_q^\times$ is the restriction of ϕ to D' and where $\alpha = \varepsilon_{2r-1} = \varepsilon_{n-1}$. Now, it is straightforward to check that

$$\xi_{\alpha, \phi(\alpha)} \xi_{\alpha, -\phi(\alpha)} = \sum_{a, b \in \mathbb{F}_q} \xi_{\varepsilon_{n-1} - \varepsilon_n, a} \xi_{\varepsilon_n, b},$$

where, for simplicity, we set $\xi_{\beta, 0} = 1_U$ for all $\beta \in \Phi^+$. Thus, by Theorem 1.1, we conclude that

$$\langle \xi_{D,\phi}, \xi_{D,\phi} \rangle = \langle \xi_{D',\phi'}, \xi_{D',\phi'} \rangle$$

and this implies that the mapping $\chi' \mapsto \chi' \xi_{\alpha, \phi(\alpha)}$ defines a one-to-one correspondence between the irreducible constituents of $\xi_{D',\phi'}$ and the irreducible constituents of $\xi_{D,\phi} = \xi_{D',\phi'} \xi_{\alpha, \phi(\alpha)}$. The above assertion follows.

Next, we prove that every irreducible character of U of maximum degree occurs as a constituent of (exactly) one of the super-characters discriminated as above. We start by proving the following result.

Proposition 6.6. *If $\chi \in \text{Irr}(U)$ has maximum degree, then $\chi(1) = q^{d(n)}$.*

Proof. Let D be the unique basic subset of Φ^+ and $\phi: D \rightarrow \mathbb{F}_q^\times$ the unique map such that $\langle \chi, \xi_{D,\phi} \rangle \neq 0$. Moreover, let $D' = D \cap \Phi_2^+$ and $\phi': D' \rightarrow \mathbb{F}_q^\times$ be the restriction of ϕ to D' . By using Proposition 2.1 (or Proposition 3.1), we see that every irreducible constituent of $\xi_{D',\phi'}$ has degree less or equal to $q^{-|D'|} \xi_{D',\phi'}(1)$. Thus,

$$\chi(1) \leq q^{-|D'|} \xi_{D,\phi}(1) = q^{-|D'|} |U : U_D|$$

(by Proposition 2.2).

On the one hand, assume that $U = UO_{2n}(q)$ and let

$$I = \{1 \leq i < n: \varepsilon_i + \varepsilon_j \notin D \text{ for all } 1 \leq j \leq n, j \neq i\}.$$

We have

$$|I| + 2|D'| = \begin{cases} n, & \text{if } \varepsilon_k + \varepsilon_n \in D \text{ for some } 1 \leq k < n, \\ n-1, & \text{if } \varepsilon_k + \varepsilon_n \notin D \text{ for all } 1 \leq k < n, \end{cases}$$

and so $|I| + |D'| \geq r$ where, either $n = 2r$, or $n = 2r + 1$. By the definition of U_D , we know that $X_{\varepsilon_i - \varepsilon_n} \subseteq U_D$ for all $i \in I$ and so $|U : U_D| \leq q^{n(n-1)/2 - |I|}$ (we recall that $A \subseteq U_D$). It follows that

$$\chi(1) \leq q^{-|D'|} |U : U_D| \leq q^{n(n-1)/2 - (|I| + |D'|)} \leq q^{n(n-1)/2 - r} = q^{d(n)},$$

as required.

On the other hand, assume that $U = UO_{2n+1}(q)$ and (similarly to above) let

$$I = \{1 \leq i \leq n: \varepsilon_i + \varepsilon_j \notin D \text{ for all } 1 \leq j \leq n, j \neq i\};$$

we note that, in this case, we have $|I| + 2|D'| = n$. By the definition of U_D , we know that $X_{\varepsilon_i} \subseteq U_D$ for all $i \in I$ and so $|U : U_D| \leq q^{n(n-1)/2 + |D'|}$ (we recall that $A \subseteq U_D$ and that $X_{\varepsilon_i} \subseteq U_D$ whenever $\varepsilon_i + \varepsilon_j \in D$ for $1 \leq i < j \leq n$). It follows that

$$\chi(1) \leq q^{-|D'|} |U : U_D| \leq q^{n(n-1)/2} = q^{d(n)},$$

as required. \square

Now, assume that $\chi \in \text{Irr}(U)$ has maximum degree $\chi(1) = q^{d(n)}$. As before, let D be the unique basic subset of Φ^+ and $\phi: D \rightarrow \mathbb{F}_q^\times$ the unique map such that χ is a constituent of the super-character $\xi_{D, \phi}$. Let $D' = D \cap \Phi_2^+$ and $I \subseteq \{1, 2, \dots, n\}$ be as in the previous proof.

Suppose that $U = UO_{2n}(q)$. Then, since $\chi(1) = q^{d(n)}$ and

$$\chi(1) \leq q^{n(n-1)/2 - (|I| + |D'|)} \leq q^{d(n)},$$

we must have $|I| + |D'| = r$ where, either $n = 2r$, or $n = 2r + 1$. This equality is possible if, and only if, either $n = 2r$, or $n = 2r + 1$ and $\varepsilon_k + \varepsilon_n \notin D$ for all $1 \leq k < n$. Firstly, suppose that the second situation occurs. Then, we must have $|D'| = r$ and so $I = \emptyset$. Thus, for each $1 \leq i < n$, there should exist $i < j_i < n$ such that $\varepsilon_i + \varepsilon_{j_i} \in D$; in particular, we deduce that $D = D' \subseteq \Phi_2^+$. On the other hand, suppose that $n = 2r$. Then, we must have

$$|D'| = \begin{cases} r, & \text{if } \varepsilon_k + \varepsilon_n \in D \text{ for some } 1 \leq k < n, \\ r-1, & \text{if } \varepsilon_k + \varepsilon_n \notin D \text{ for all } 1 \leq k < n. \end{cases}$$

If $\varepsilon_k + \varepsilon_n \in D$ for some $1 \leq k < n$, then $I = \emptyset$ and so $D = D' \subseteq \Phi_2^+$. Otherwise, suppose that $\varepsilon_k + \varepsilon_n \notin D$ for all $1 \leq k < n$. Then, $|I| = 1$ and so, there exists $1 \leq k < n$ such that

$X_{\varepsilon_k - \varepsilon_n} \subseteq U_D$. If $\varepsilon_k - \varepsilon_n \notin D$ and $k < n - 1$, then $X_{\varepsilon_k - \varepsilon_{n-1}} \subseteq U_D$ (by the definition of U_D) and this implies that

$$\chi(1) \leq q^{r-1} |U : U_D| \leq q^{n(n-1)/2-r-1} = q^{d(n)-1},$$

a contradiction. It follows that, either $I = \{n-1\}$ and $D \subseteq D' \cup \{\varepsilon_{n-1} - \varepsilon_n\}$, or there exists $1 \leq k < n-1$ such that $I = \{k\}$ and $D = D' \cup \{\varepsilon_k - \varepsilon_n\}$. Further, in any case, for each $1 \leq i < n$ with $i \notin I$, there should exist $1 \leq j_i < n$, $j_i \notin I \cup \{i\}$, such that $\varepsilon_i + \varepsilon_{j_i} \in D$.

Now, suppose that $U = UO_{2n+1}(q)$. Firstly, assume that there exist $i, j \in I$ with $1 \leq i \neq j < n$. Then, by the definition of U_D , it is clear that at least one of $X_{\varepsilon_i - \varepsilon_n}$ or $X_{\varepsilon_j - \varepsilon_n}$ is a subgroup of U_D . Thus, we must have

$$\chi(1) \leq q^{n(n-1)/2-1} < q^{n(n-1)/2} = q^{d(n)},$$

a contradiction. It follows that, either $|I| \leq 1$, or $I = \{i, n\}$ for some $1 \leq i < n$. Further, if $i \in I$ for some $1 \leq i < n$, then we must have $\varepsilon_i \in D$; otherwise, $X_{\varepsilon_i - \varepsilon_n} \subseteq U_D$ and so $\chi(1) \leq q^{n(n-1)/2-1}$, a contradiction. Therefore, we conclude that one of the following cases must occur (we recall that $|I| + 2|D'| = n$).

- (i) $I = \emptyset$; hence, n must be even, $|D'| = n/2$ and, for each $1 \leq i \leq n$, there exists $1 \leq j_i \leq n$, $j_i \neq i$, such that $\varepsilon_i + \varepsilon_{j_i} \in D$.
- (ii) $I = \{i\}$ for some $1 \leq i \leq n$ with $\varepsilon_i \in D$; hence, n must be odd, $|D'| = (n-1)/2$ and, for each $1 \leq k \leq n$ with $k \neq i$, there should exist $1 \leq j_k \leq n$, $j_k \neq i, k$, such that $\varepsilon_k + \varepsilon_{j_k} \in D$.
- (iii) $I = \{i, n\}$ for some $1 \leq i < n$ with $\varepsilon_i \in D$; hence, n must be even, $|D'| = (n-2)/2$ and, for each $1 \leq k < n$ with $k \neq i$, there should exist $1 \leq j_k < n$, $j_k \neq i, k$, such that $\varepsilon_k + \varepsilon_{j_k} \in D$ (we note that $\varepsilon_k + \varepsilon_n \notin D$ for all $1 \leq k \leq n$).

We are now able to prove the main result of this subsection.

Theorem 6.7. Let $\chi \in \text{Irr}(U)$ have maximum degree $\chi(1) = q^{d(n)}$, let D be the unique basic subset of Φ^+ and let $\phi : D \rightarrow \mathbb{F}_q^\times$ the unique map such that $\langle \chi, \xi_{D, \phi} \rangle \neq 0$. Then, the following holds.

- (i) If $U = UO_{2n}(q)$, then:
 - (a) if n is even, $D = \{\varepsilon_1 + \varepsilon_2, \varepsilon_3 + \varepsilon_4, \dots, \varepsilon_{n-3} + \varepsilon_{n-2}\} \cup D_1$ where $D_1 \subsetneq \{\varepsilon_{n-1} + \varepsilon_n, \varepsilon_{n-1} - \varepsilon_n\}$;
 - (b) if n is odd, $D = \{\varepsilon_1 + \varepsilon_2, \varepsilon_3 + \varepsilon_4, \dots, \varepsilon_{n-1} + \varepsilon_n\}$.
- (ii) If $U = UO_{2n+1}(q)$, then:
 - (a) if n is even, either $D = \{\varepsilon_1 + \varepsilon_2, \varepsilon_3 + \varepsilon_4, \dots, \varepsilon_{n-1} + \varepsilon_n\}$, or $D = \{\varepsilon_1 + \varepsilon_2, \varepsilon_3 + \varepsilon_4, \dots, \varepsilon_{n-3} + \varepsilon_{n-2}\} \cup \{\varepsilon_{n-1}\}$;
 - (b) if n is odd, either $D = \{\varepsilon_1 + \varepsilon_2, \varepsilon_3 + \varepsilon_4, \dots, \varepsilon_{n-2} + \varepsilon_{n-1}\}$, or $D = \{\varepsilon_1 + \varepsilon_2, \varepsilon_3 + \varepsilon_4, \dots, \varepsilon_{n-2} + \varepsilon_{n-1}\} \cup \{\varepsilon_n\}$.

Proof. We proceed by induction on n . Since the result is clear if $n \leq 2$, we assume that $n > 2$ and that the theorem holds for all integers less than n .

On the one hand, assume that $\varepsilon_1 + \varepsilon_2 \in D$. Let $D_0 = D - \{\varepsilon_1 + \varepsilon_2\}$, let $\phi_0 : D_0 \rightarrow \mathbb{F}_q^\times$ be the restriction of ϕ to D_0 and consider the super-character ξ_{D_0, ϕ_0} of U . Let U_0 be the subgroup of

U consisting of all matrices $x \in U$ which satisfy $x_{a,b} = 0$ for all $(a, b) \in \mathcal{E}$ with $a \in \{1, 2\}$. It is clear that

$$U_0 \cong \begin{cases} UO_{2(n-2)}, & \text{if } U = UO_{2n}(q), \\ UO_{2(n-2)+1}, & \text{if } U = UO_{2n+1}(q), \end{cases}$$

and that every irreducible constituent of ξ_{D_0, ϕ_0} is naturally identified with an irreducible character of U_0 . Since

$$\xi_{D, \phi} = \xi_{\varepsilon_1 + \varepsilon_2, \phi(\varepsilon_1 + \varepsilon_2)} \xi_{D_0, \phi_0},$$

χ is an irreducible constituent of $\tau\chi_0$ where $\tau \in \text{Irr}(U)$ is an irreducible constituent of $\xi_{\varepsilon_1 + \varepsilon_2, \phi(\varepsilon_1 + \varepsilon_2)}$ and $\chi_0 \in \text{Irr}(U)$ is an irreducible constituent of ξ_{D_0, ϕ_0} . By Proposition 2.1 (or Proposition 3.2), we have

$$\tau(1) = q^{-1} \xi_{\varepsilon_1 + \varepsilon_2, \phi(\varepsilon_1 + \varepsilon_2)}(1) = \begin{cases} q^{2(n-2)}, & \text{if } U = UO_{2n}(q), \\ q^{(n-1)+(n-2)}, & \text{if } U = UO_{2n+1}(q), \end{cases}$$

and, by the previous proposition, we know that $\chi_0(1) \leq q^{d(n-2)}$. In any case, we conclude that $(\tau\chi_0)(1) \leq q^{d(n)}$ and, since $\chi(1) = q^{d(n)}$, we must have $\chi = \tau\chi_0$. In particular, we obtain $\chi_0(1) = q^{d(n-2)}$ and so, by induction, we deduce that

- (i) If $U = UO_{2n}(q)$, then:
 - (a) if n is even, $D_0 = \{\varepsilon_3 + \varepsilon_4, \varepsilon_5 + \varepsilon_6, \dots, \varepsilon_{n-3} + \varepsilon_{n-2}\} \cup D_1$ where $D_1 \subsetneq \{\varepsilon_{n-1} + \varepsilon_n, \varepsilon_{n-1} - \varepsilon_n\}$;
 - (b) if n is odd, $D_0 = \{\varepsilon_3 + \varepsilon_4, \varepsilon_5 + \varepsilon_6, \dots, \varepsilon_{n-1} + \varepsilon_n\}$.
- (ii) If $U = UO_{2n+1}(q)$, then:
 - (a) if n is even, either $D_0 = \{\varepsilon_3 + \varepsilon_4, \varepsilon_5 + \varepsilon_6, \dots, \varepsilon_{n-1} + \varepsilon_n\}$, or $D_0 = \{\varepsilon_3 + \varepsilon_4, \varepsilon_5 + \varepsilon_6, \dots, \varepsilon_{n-3} + \varepsilon_{n-2}\} \cup \{\varepsilon_{n-1}\}$;
 - (b) if n is odd, either $D_0 = \{\varepsilon_3 + \varepsilon_4, \varepsilon_5 + \varepsilon_6, \dots, \varepsilon_{n-2} + \varepsilon_{n-1}\}$, or $D_0 = \{\varepsilon_3 + \varepsilon_4, \varepsilon_5 + \varepsilon_6, \dots, \varepsilon_{n-2} + \varepsilon_{n-1}\} \cup \{\varepsilon_n\}$.

Thus, in the case where $\varepsilon_1 + \varepsilon_2 \in D$, we conclude that $D = D_0 \cup \{\varepsilon_1 + \varepsilon_2\}$ is as required.

On the other hand, assume that $\varepsilon_1 + \varepsilon_2 \notin D$. We start by considering the case where $D \not\subseteq \Phi_2^+$. By the discussion preceding the theorem, we know that one of the following cases must occur.

- (i) $U = UO_{2n}(q)$ with n even, and

$$D = (D \cap \Phi_2^+) \cup \{\varepsilon_i - \varepsilon_n\}, \quad \text{for some } 1 \leq i < n;$$

moreover, we have $|D \cap \Phi_2^+| = (n-2)/2$ and, for each $2 \leq k < n$, there exists $1 \leq j_k < n$, $j_k \neq i, k$, such that $\varepsilon_k + \varepsilon_{j_k} \in D$ (we note also that $\varepsilon_k + \varepsilon_n \notin D$ for all $1 \leq k < n$).

- (ii) $U = UO_{2n+1}(q)$ with n odd, and

$$D = (D \cap \Phi_2^+) \cup \{\varepsilon_i\}, \quad \text{for some } 1 \leq i \leq n;$$

moreover, we have $|D \cap \Phi_2^+| = (n-1)/2$ and, for each $1 \leq k \leq n$ with $k \neq i$, there exists $1 \leq j_k \leq n$, $j_k \neq i, k$, such that $\varepsilon_k + \varepsilon_{j_k} \in D$.

(iii) $U = UO_{2n+1}(q)$ with n even, and

$$D = (D \cap \Phi_2^+) \cup \{\varepsilon_i\}, \quad \text{for some } 1 \leq i < n;$$

moreover, we have $|D \cap \Phi_2^+| = (n-2)/2$ and, for each $1 \leq k < n$ with $k \neq i$, there exists $1 \leq j_k < n$, $j_k \neq i, k$, such that $\varepsilon_k + \varepsilon_{j_k} \in D$ (we note also that $\varepsilon_k + \varepsilon_n \notin D$ for all $1 \leq k < n$).

In any case, suppose that, either $i < n-1$, or $i = n-1$ and there exists $1 \leq j < n$ such that $\varepsilon_j + \varepsilon_n \in D$ (we note that in this situation occurs only in the case (2)). Let

$$\alpha = \begin{cases} \varepsilon_i - \varepsilon_n, & \text{if } U = UO_{2n}(q) \text{ (with } n \text{ even),} \\ \varepsilon_i, & \text{if } U = UO_{2n+1}(q). \end{cases}$$

Then, we have

$$\xi_{D,\phi} = \xi_{\alpha,\phi(\alpha)} \prod_{\beta \in D \cap \Phi_2^+} \xi_{\beta,\phi(\beta)}$$

(by the definition). Let $1 \leq k < l \leq n$ be such that $\beta = \varepsilon_k + \varepsilon_l \in D$ and, for each $s \in \mathbb{F}_q$, let $\tau_{\beta,s}$ denote the irreducible character of U corresponding (under Kazhdan's correspondence) to the coadjoint U -orbit which contain the element

$$f_{\beta,s} = \phi(\beta)e_\beta^* + se_{\varepsilon_k - \varepsilon_l}^* \in \mathfrak{u}^*.$$

By Proposition 3.2, the elementary character $\xi_{\beta,\phi(\beta)}$ is multiplicity free and we have

$$\xi_{\beta,\phi(\beta)} = \sum_{s \in \mathbb{F}_q} \tau_{\beta,s}.$$

It follows that there exists a map $\psi: D \cap \Phi_2^+ \rightarrow \mathbb{F}_q$ such that χ is an irreducible constituent of the character

$$\zeta = \xi_{\alpha,\phi(\alpha)} \prod_{\beta \in D \cap \Phi_2^+} \tau_{\beta,\psi(\beta)}.$$

Now, it is easy to check that

$$\zeta(1) = q^{-|D \cap \Phi_2^+|} \xi_{D,\phi}(1) \leq q^{d(n)}$$

and, since $\chi(1) = q^{d(n)}$, we conclude that $\chi = \zeta$. Thus, by [1, Corollary 1] (and by Kazhdan's correspondence), the irreducible character χ corresponds to the U -orbit $\mathcal{O}(f) \subseteq \mathfrak{u}^*$ which contains the element

$$f = \phi(\alpha)e_\alpha^* + \sum_{\beta \in D \cap \Phi_2^+} f_{\beta,\psi(\beta)}.$$

Since $|\mathcal{O}(f)| = \chi(1)^2 = q^{2d(n)}$, Proposition 3.1 implies that

$$\mathfrak{p} = \mathbb{F}_q e_\alpha + \sum_{\substack{1 \leq k < l \leq n \\ \varepsilon_k + \varepsilon_l \in D}} (\mathbf{u}_{\varepsilon_k + \varepsilon_l} + \mathbb{F}_q e_{\varepsilon_k - \varepsilon_l})$$

is a maximal f -isotropic subspace of \mathfrak{u} (see Section 3 for the definition of $\mathbf{u}_{\varepsilon_k + \varepsilon_l}$). Finally, let $1 \leq j \leq n$ be such that $\varepsilon_{i+1} + \varepsilon_j \in D$. Then, it is easy to check that the vector

$$u = \begin{cases} e_{\varepsilon_i - \varepsilon_{i+1}} - \phi(\varepsilon_i - \varepsilon_n)^{-1} \phi(\varepsilon_{i+1} + \varepsilon_j) e_{\varepsilon_j + \varepsilon_n}, & \text{if } U = UO_{2n}(q), \\ e_{\varepsilon_i - \varepsilon_{i+1}} - \phi(\varepsilon_i - \varepsilon_n)^{-1} \phi(\varepsilon_{i+1} + \varepsilon_j) e_{\varepsilon_j}, & \text{if } U = UO_{2n+1}(q), \end{cases}$$

lies in $\tau \mathfrak{ad}(f)$ and so $\mathfrak{p} + \mathbb{F}_q u$ is an f -isotropic subspace, contradicting the maximality of \mathfrak{p} (we note that $u \notin \mathfrak{p}$). This contradiction implies that, either $i = n - 1$ and $\varepsilon_k + \varepsilon_n \notin D$ for all $1 \leq k < n$, or $i = n$ (and, if this is the case, then $U = UO_{2n+1}(q)$ with n odd). In any case, since $\varepsilon_1 + \varepsilon_2 \notin D$, there should exist $2 < j' \leq n$ such that $\varepsilon_1 + \varepsilon_j, \varepsilon_2 + \varepsilon_{j'} \in D$. In what follows, we assume that this situation occurs and we observe that our argument is also valid in the case where $D \subseteq \Phi_2^+$.

For each $\alpha \in D \cap \Phi_2^+$ and each $s \in \mathbb{F}_q$, we define $f_{\alpha,s} \in \mathfrak{u}^*$ and $\tau_{\alpha,s} \in \text{Irr}(U)$ as above, and we extend this notation to any root $\alpha \in D - \Phi_2^+$ (if it exists) by setting $f_{\alpha,s} = s e_\alpha^*$ and $\tau_{\alpha,s} = \xi_{\alpha,s}$; for convenience, we set $\xi_{\alpha,0} = 1_U$. Then, since

$$\xi_{D,\phi} = \prod_{\alpha \in D} \xi_{\alpha,\phi(\alpha)}$$

(by the definition), the argument above may be repeated to prove that there exists a map $\psi : D \rightarrow \mathbb{F}_q$ such that

$$\chi = \prod_{\alpha \in D} \tau_{\alpha,\psi(\alpha)};$$

we observe that, for $\alpha \in D - \Phi_2^+$, we must have $\psi(\alpha) = \phi(\alpha) \in \mathbb{F}_q^\times$. Then, by [1, Corollary 1] (and by Kazhdan's correspondence), χ corresponds to the U -orbit $\mathcal{O}(f) \subseteq \mathfrak{u}^*$ which contains the element

$$f = \sum_{\alpha \in D} f_{\alpha,\psi(\alpha)}.$$

Since $|\mathcal{O}(f)| = \chi(1)^2 = q^{2d(n)}$, we easily conclude that

$$\mathfrak{p} = \sum_{\alpha \in D - \Phi_2^+} \mathbb{F}_q e_\alpha + \sum_{\substack{1 \leq k < l \leq n \\ \varepsilon_k + \varepsilon_l \in D}} (\mathbf{u}_{\varepsilon_k + \varepsilon_l} + \mathbb{F}_q e_{\varepsilon_k - \varepsilon_l})$$

is a maximal f -isotropic subspace of \mathfrak{u} . However, the vector

$$u = \begin{cases} e_{\varepsilon_2 - \varepsilon_j} - \phi(\varepsilon_1 + \varepsilon_j)^{-1} \phi(\varepsilon_2 + \varepsilon_{j'}) e_{\varepsilon_1 - \varepsilon_{j'}}, & \text{if } j < j', \\ e_{\varepsilon_1 - \varepsilon_2} - \phi(\varepsilon_2 + \varepsilon_{j'})^{-1} \phi(\varepsilon_1 + \varepsilon_j) e_{\varepsilon_{j'} - \varepsilon_j}, & \text{if } j' < j, \end{cases}$$

lies in $\text{rad}(f)$ and so $\mathfrak{p} + \mathbb{F}_q u$ is an f -isotropic subspace. This contradiction implies that $\varepsilon_1 + \varepsilon_2 \in D$ and this completes the proof of the theorem. \square

References

- [1] C.A.M. André, Basic characters of the unitriangular group, *J. Algebra* 175 (1995) 287–319.
- [2] C.A.M. André, Basic sums of coadjoint orbits of the unitriangular group, *J. Algebra* 176 (1995) 959–1000.
- [3] C.A.M. André, Basic characters of the unitriangular group (for arbitrary primes), *Proc. Amer. Math. Soc.* 130 (7) (2002) 1943–1954.
- [4] C.A.M. André, Hecke algebras for the basic characters of the unitriangular group, *Proc. Amer. Math. Soc.* 132 (4) (2003) 987–996.
- [5] E. Artin, *Geometric Algebra*, Interscience, New York, 1957.
- [6] R.W. Carter, *Simple Groups of Lie Type*, Wiley, London, 1972.
- [7] R.W. Carter, *Finite Groups of Lie Type (Conjugacy Classes and Complex Characters)*, Wiley, New York, 1985.
- [8] P.M. Cohn, *Algebra*, vol. 1, second ed., Wiley, New York, 1988.
- [9] L. Corwin, F.P. Greenleaf, *Representations of Nilpotent Lie Groups and Their Applications. Part 1: Basic Theory and Examples*, Cambridge Stud. Adv. Math., vol. 18, Cambridge Univ. Press, Cambridge, 1990.
- [10] C.W. Curtis, I. Reiner, *Methods of Representation Theory (with applications to finite groups and orders)*, vol. 2, Wiley, New York, 1987.
- [11] E. Arias-Castro, P. Diaconis, R. Stanley, A super-class walk on upper-triangular matrices, *J. Algebra* 278 (2004) 739–765.
- [12] J. Dixmier, *Enveloping Algebras*, Grad. Stud. Math., vol. 11, Amer. Math. Soc., Providence, RI, 1996.
- [13] I.M. Isaacs, *Character Theory of Finite Groups*, Dover, New York, 1994.
- [14] N. Jacobson, *Lie Algebras*, Dover, New York, 1979.
- [15] D. Kazhdan, Proof of Springer’s hypothesis, *Israel J. Math.* 28 (1977) 272–286.
- [16] A. Kirillov, *Elements of the Theory of Representations*, Springer-Verlag, New York, 1976.
- [17] A. Kirillov, *Lectures on the Orbit Method*, Grad. Stud. Math., vol. 64, Amer. Math. Soc., Providence, RI, 2004.
- [18] S. Mukherjee, Coadjoint orbits for A_{n-1}^+ , B_n^+ and D_n^+ , available from: <http://arXiv.org/pdf/math.RT/0501332>.
- [19] A.M. Neto, Caracteres e órbitas coadjuntas de p -grupos lineares finitos, PhD thesis, Departamento de Matemática, Faculdade de Ciências da Universidade de Lisboa, 2006.
- [20] B. Srinivasan, *Representations of Finite Chevalley Groups*, Lecture Notes in Math., vol. 366, Springer-Verlag, New York, 1974.